

# CONSTRUCTING INFINITE COMATRIX CORINGS FROM COLIMITS

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**ABSTRACT.** We propose a class of infinite comatrix corings, and describe them as colimits of systems of usual comatrix corings. The infinite comatrix corings of El Kaoutit and Gómez Torrecillas are special cases of our construction, which in turn can be considered as a special case of the comatrix corings introduced recently by Gómez Torrecillas and the third author.

## INTRODUCTION

Corings were introduced by Sweedler in 1975 [15]; since the beginning of the century, there has been a renewed interest in corings, initiated by an observation made by Takeuchi that most type of modules that are considered in Hopf algebra theory, like Hopf modules, Yetter-Drinfeld modules, entwined modules, are in fact comodules over certain corings. A detailed discussion of recent applications of corings can be found in [6].

One of the beautiful applications is a reformulation of descent theory and Galois theory. To a ring morphism  $B \rightarrow A$ , we can associate a coring  $A \otimes_B A$ , called Sweedler's canonical coring, and the category of descent data is isomorphic to the category of comodules over the coring. To an action or coaction of a group or Hopf algebra on  $A$ , we can associate a coring, and there exists a canonical coring map from Sweedler's coring to this coring. A necessary condition for the Galois descent is that this map is an isomorphism. This was observed by Brzeziński in his paper [4], see also [8] for a detailed discussion.

A more general theory was proposed by El Kaoutit and Gómez Torrecillas [10]. We start from two rings  $A$  and  $B$ , connected by a  $(B, A)$ -module  $P$ . If  $P$  is finitely generated and projective as a right  $A$ -module,  $P^* \otimes_B P$  is an  $A$ -coring. If  $A$  and  $B$  are connected via a ring morphism  $B \rightarrow A$ , then we can take  $P = A$  considered as a  $(B, A)$ -bimodule, and we recover Sweedler's coring.  $P^* \otimes_B P$  is called a comatrix coring, and several properties of the theory outlined in [8] can be generalized, we refer to [9] and [10].

The condition that  $P$  is finitely generated and projective as a right  $A$ -module is crucial in the theory. Nevertheless, El Kaoutit and Gómez Torrecillas [11] proposed an infinite version of comatrix corings, starting from an infinite collection of finitely generated projective right  $A$ -modules  $\{P_i \mid i \in I\}$ . They consider the direct sum  $P$  of the  $P_i$ , and the direct sum  $P^\dagger$  of the  $P_i^*$ . The tensor product of  $P^\dagger$  and  $P$  over a suitable ring  $R$  is then a coring, called the infinite comatrix coring. They give several descriptions and properties of this coring, including a version of the Faithfully Flat Descent Theorem. One of the important features is the fact that the ring  $R$  has no unit; it is a ring with orthogonal idempotent local units.

The natural framework needed to introduce infinite comatrix corings was proposed recently by Gómez Torrecillas and the third author in [12]. The philosophy is the following. Let  $P$  be a  $(B, A)$ -bimodule, and consider the functor  $F = - \otimes_B P : \mathcal{M}_B \rightarrow \mathcal{M}_A$ .  $F$  has a right adjoint  $G$  of the form  $- \otimes_A Q$  for some  $(B, A)$ -bimodule  $Q$  if and only if  $P$  is finitely generated and projective as a right  $A$ -module, and in this case  $Q \cong P^*$ . Then we have a so-called comatrix coring context (see [5] or [7] for the definition). In fact we need such an adjunction to be able to define the coproduct on the comatrix coring. Instead of considering rings with a unit, we now consider firm rings, these are rings  $A$  having the property that the canonical map  $A \otimes_A A \rightarrow A$  is an isomorphism. Firm bimodules over firm rings form a monoidal category, and we can consider corings over firm rings. The bimodules in a comatrix coring context connecting firm rings are not necessarily finitely generated projective. The comatrix coring contexts from [11] are of this type: only one of the two rings involved has units, the other one has only local units (a complete set of orthogonal idempotents).

In this paper, we propose some classes of infinite comatrix corings. In the first three Sections, we have collected some necessary preliminary results: in Section 1, we briefly introduce the comatrix corings from [12], and recall some of the elementary results, for example the Faithfully Flat Descent Theorem; in Section 2 we show that the colimit of a functor that is a coalgebra in a functor category is itself a coalgebra; in Section 3, we discuss split directed systems and their colimits. The main results appear in Section 4: we describe comatrix corings associated to an  $(A, B)$ -bimodule  $P$ , where  $A$  is a ring with unit, and  $B$  a ring with idempotent local units. These rings are colimits of a (split) directed system of rings with unit and we can describe the category of firm  $B$ -modules. The comatrix corings can also be described as colimits. Firm modules over rings with idempotent local units are constructed in Section 5: we consider a split direct system  $\underline{M}$  in some  $k$ -linear category  $\mathcal{A}$  with a colimit, and a product preserving functor  $\omega$  to the category of right  $A$ -modules. The ring  $B$  is then the colimit of the  $\mathcal{A}$ -endomorphism rings of the  $M_i$ , and the  $\omega(M_i) = P_i$  form a split direct system of  $(B_i, A)$ -bimodules. An interesting special case is considered in Section 6: we consider an  $A$ -coring  $\mathcal{C}$ , and let  $\mathcal{A} = \mathcal{M}^{\mathcal{C}}$ , and  $\omega$  the functor forgetting the coaction. In this situation, we can define a canonical coring map from the associated comatrix coring to  $\mathcal{D}$ , and  $\underline{M}$  is called a system of Galois  $\mathcal{C}$ -comodules if this map is an isomorphism. The comatrix corings introduced in [11] are special cases.

## 1. COMATRIX CORINGS OVER FIRM ALGEBRAS

Let  $k$  be a commutative ring, and  $A$  a  $k$ -algebra, not necessarily with a unit. If  $A$  has a unit, then the canonical map  $A \otimes_A A \rightarrow A$  is an isomorphism, but not conversely. We say that  $A$  is a *firm algebra* if  $A \otimes_A A \rightarrow A$  is an isomorphism. In [16], firm algebras are called *regular algebras*; in [7], they are called *unital*. Algebras with local units are firm.

Let  $A$  be a firm algebra. A right  $A$ -module is called *firm* if the canonical map  $M \otimes_A A \rightarrow M$  is an isomorphism. If  $A$  is an algebra with unit, then all modules are firm.  $\mathcal{M}_A$  will be the category of *firm* right  $A$ -modules and right  $A$ -linear maps. In a similar way, we introduce the categories of firm left modules and firm bimodules. The category of all (not necessarily firm) right  $A$ -modules will be denoted by  $\widetilde{\mathcal{M}}_A$ . A (firm) left  $A$ -module is called *flat* if the functor  $- \otimes_A M : \widetilde{\mathcal{M}}_A \rightarrow \underline{\mathbf{Ab}}$  is exact.

The category  $\mathcal{M}_A$  of firm right  $A$ -modules over a firm ring  $A$  is always an abelian category. Under the extra condition that  $A$  is flat as left  $A$ -module, the kernels in  $\mathcal{M}_A$  can be computed already in Ab. If  $A$  is a ring with local units, i.e. for every  $a \in A$ , there exists an  $e \in A$  such that  $ae = ea = a$ , then  $A$  is a firm ring and  $A$  is flat as a left and right  $A$ -module ( $A$  is even locally projective as a left and right  $A$ -module), so in this situation the kernels of  $\mathcal{M}_A$  and  ${}_A\mathcal{M}$  can be computed in Ab.

The category  ${}_A\mathcal{M}_A$  of firm  $A$ -bimodules is a monoidal category, so we can consider corings over firm algebras, these are coalgebras in the monoidal category  ${}_A\mathcal{M}_A$ . If  $\mathcal{C}$  is a coring over a firm  $k$ -algebra  $A$ , then we can define left and right  $\mathcal{C}$ -comodules. A right  $\mathcal{C}$ -comodule  $(M, \rho^r)$  is a firm right  $A$ -module together with a right  $A$ -linear map  $\rho^r : M \rightarrow M \otimes_A \mathcal{C}$  satisfying the usual coassociativity and counit properties. The category of right  $\mathcal{C}$ -comodules and  $\mathcal{C}$ -colinear maps is denoted by  $\mathcal{M}^{\mathcal{C}}$ . Similarly one introduces categories  ${}^{\mathcal{C}}\mathcal{M}$ ,  ${}_B\mathcal{M}^{\mathcal{C}}$  and  ${}^{\mathcal{C}}\mathcal{M}^{\mathcal{C}}$ . For  $M \in {}^{\mathcal{C}}\mathcal{M}$  and  $N \in \mathcal{M}^{\mathcal{C}}$ , we define the cotensor product  $N \otimes^{\mathcal{C}} M$  as the following equalizer in Ab:

$$(1) \quad N \otimes^{\mathcal{C}} M \longrightarrow N \otimes_A M \xrightleftharpoons[\rho^r \otimes_A M]{N \otimes_A \rho^l} N \otimes_A \mathcal{C} \otimes_A M.$$

If  $M \in {}^{\mathcal{C}}\mathcal{M}_B$ , where  $B$  is a firm ring, which is flat as a left  $B$ -module, then (1) is also an equalizer in  $\mathcal{M}_B$ , hence  $N \otimes^{\mathcal{C}} M$  is a firm right  $B$ -module.

Let  $A$  and  $B$  be firm  $k$ -algebras. The notion of Morita context can be generalized, by requiring that the connecting bimodules are firm. If one of the morphisms in a Morita context is bijective, then we can associate a pair of adjoint functors to it. More generally, we have the following result (see [7, Theorem 1.1.3]).

**Proposition 1.1.** *Let  $B$  and  $A$  be firm  $k$ -algebras, and  $P \in {}_B\mathcal{M}_A$ ,  $P^\dagger \in {}_A\mathcal{M}_B$  firm bimodules. Consider two bimodule maps*

$$\eta : B \rightarrow P \otimes_A P^\dagger \text{ and } \varepsilon : P^\dagger \otimes_B P \rightarrow A.$$

*We use the following Sweedler-type notation:*

$$\eta(b) = b^- \otimes_A b^+ \in P \otimes_A P^\dagger,$$

*where summation is implicitly understood, as usual. Assume that  $\eta$  and  $\varepsilon$  satisfy the following formulas, for all  $b \in B$ ,  $p \in P$ ,  $q \in P^\dagger$ :*

$$(2) \quad b^- \varepsilon(b^+ \otimes_A p) = bp ; \varepsilon(q \otimes_A b^-) b^+ = qb.$$

*Then we have a pair of adjoint functors  $(F, G)$*

$$F = - \otimes_B P : \mathcal{M}_B \rightarrow \mathcal{M}_A ; G = - \otimes_A P^\dagger : \mathcal{M}_A \rightarrow \mathcal{M}_B.$$

*Proof.* The unit and counit of the adjunction are

$$\eta_N = N \otimes_B \eta : N \rightarrow N \otimes_B P \otimes_A P^\dagger ; \varepsilon_M = M \otimes_A \varepsilon : M \otimes_A P^\dagger \otimes_B P \rightarrow M,$$

for all  $N \in \mathcal{M}_B$ ,  $M \in \mathcal{M}_A$ . □

Following [5],  $(B, A, P, P^\dagger, \eta, \varepsilon)$  is called a *comatrix coring context*. To a comatrix coring context  $(B, A, P, P^\dagger, \eta, \varepsilon)$ , we can associate an  $A$ -coring  $\mathcal{D}$  (called comatrix coring) and a  $B$ -ring  $\mathcal{A}$  (called matrix ring, or elementary algebra (see [7, 16])). They are given by the following data:

$\mathcal{D} = P^\dagger \otimes_B P$ , with

$$\Delta_{\mathcal{D}} = P^\dagger \otimes_B \eta \otimes_B P : \mathcal{D} \rightarrow \mathcal{D} \otimes_A \mathcal{D} ; \varepsilon_{\mathcal{D}} = \varepsilon : \mathcal{D} \rightarrow A.$$

$\mathcal{A} = P \otimes_A P^\dagger$ , with

$$m_{\mathcal{A}} = P \otimes_A \varepsilon \otimes_A P^\dagger : \mathcal{A} \otimes_B \mathcal{A} \rightarrow \mathcal{A} ; \eta_{\mathcal{A}} = \eta : B \rightarrow P \otimes_A P^\dagger.$$

$P$  is a right  $\mathcal{D}$ -comodule, and  $P^\dagger$  is a left  $\mathcal{D}$ -comodule; the right and left coactions are the following:

$$\begin{aligned} \rho^r : P &\rightarrow P \otimes_A P^\dagger \otimes_B P, \quad \rho^r(bp) = b^- \otimes_A b^+ \otimes_B p \\ \rho^l : P^\dagger &\rightarrow P^\dagger \otimes_B P \otimes_A P^\dagger, \quad \rho^l(qb) = q \otimes_B b^- \otimes_A b^+ \end{aligned}$$

**Proposition 1.2.** *Let  $(B, A, P, P^\dagger, \eta, \varepsilon)$  be a comatrix coring context, and assume that  $B$  is flat as left  $B$ -module. Then we have a pair of adjoint functors  $(K, R)$*

$$K = - \otimes_B P : \mathcal{M}_B \rightarrow \mathcal{M}^{\mathcal{D}} ; R = - \otimes^{\mathcal{D}} P^\dagger : \mathcal{M}^{\mathcal{D}} \rightarrow \mathcal{M}_B.$$

*Proof.* It follows from the comments preceeding Proposition 1.1 that  $R(M)$  is a firm right  $B$ -module, for every  $M \in \mathcal{M}^{\mathcal{D}}$ . We restrict to giving the unit and the counit of the adjunction. For  $N \in \mathcal{M}_B$  and  $M \in \mathcal{M}^{\mathcal{D}}$ , we have

$$\begin{aligned} \eta_N : N &\rightarrow (N \otimes_B P) \otimes^{\mathcal{D}} P^\dagger, \quad \eta_N(nb) = n \otimes_B b^- \otimes_A b^+ ; \\ \varepsilon_M : (M \otimes^{\mathcal{D}} P^\dagger) \otimes_B P &\rightarrow M, \quad \varepsilon_M\left(\sum_j m_j \otimes_A q_j \otimes p_j\right) = \sum_j m_j \varepsilon(q_j \otimes_B p_j). \end{aligned}$$

Let us show that  $\eta_N(n) \in (N \otimes_B P) \otimes^{\mathcal{D}} P^\dagger$ , for all  $n \in N$ . Since  $N$  is firm as a right  $B$ -module, it suffices to look at elements of the form  $nbcd$ , with  $n \in N$ ,  $b, c, d \in B$ . Since  $\eta$  is a  $B$ -bimodule map, we have, for all  $b, c \in B$  that  $\eta(bc) = b\eta(c) = \eta(b)c$ , or

$$(3) \quad (bc)^- \otimes_A (bc)^+ = bc^- \otimes_A c^+ = b^- \otimes_A b^+ c.$$

Using (3), we find easily that

$$\begin{aligned} (\rho_{N \otimes_B P}^r \otimes_A P^\dagger)(\eta_N(nbcd)) &= (\rho_{N \otimes_B P}^r \otimes_A P^\dagger)(nb \otimes_B c^- \otimes_A c^+ d) \\ &= (N \otimes \rho^r \otimes P^\dagger)(n \otimes_B bcd^- \otimes_A d^+) \\ &= n \otimes_B b^- \otimes_A b^+ \otimes_B cd^- \otimes_A d^+ \\ &= n \otimes_B b^- \otimes_A b^+ c \otimes_B d^- \otimes_A d^+ \\ &= nb \otimes_B c^- \otimes_A c^+ \otimes_B d^- \otimes_A d^+ \\ &= ((N \otimes_B P) \otimes_A \rho^l)(nb \otimes_B c^- \otimes_A c^+ d) \\ &= ((N \otimes_B P) \otimes_A \rho^l)(\eta_N(nbcd)). \end{aligned}$$

□

**Theorem 1.3. (Faithfully flat descent)** *Let  $(B, A, P, P^\dagger, f, g)$  be a comatrix coring context, and assume that  $B$  and  $P$  are flat as a left  $B$ -module. Then  $R$  is fully faithful.  $(K, R)$  is a pair of inverse equivalences if and only if  $P$  is faithfully flat as a left  $B$ -module.*

*Proof.* Take  $M \in \mathcal{M}^{\mathcal{D}}$ . If  $P \in {}_B\mathcal{M}$  is flat, then the map

$$j : (M \otimes^{\mathcal{D}} P^\dagger) \otimes_B P \rightarrow M \otimes^{\mathcal{D}} (P^\dagger \otimes_B P), \quad j\left(\left(\sum_i m_i \otimes_A p_i\right) \otimes_B q\right) = \sum_i m_i \otimes_A (p_i \otimes_B q)$$

is an isomorphism. The map

$$M \otimes_A \varepsilon : M \otimes^{\mathcal{D}} (P^\dagger \otimes_B P) \rightarrow M \otimes_A A \cong M$$

is an isomorphism. If  $P \in {}_B\mathcal{M}$  is flat, then  $\varepsilon_M = (M \otimes_A \varepsilon) \circ j$  is an isomorphism. Assume that  $P \in {}_B\mathcal{M}$  is flat. We have to show that  $\eta_N$  is an isomorphism, for every  $N \in \mathcal{M}_B$ . It suffices to show that the sequence

$$S : 0 \longrightarrow N \xrightarrow{N \otimes_B \eta} N \otimes_B P \otimes_A P^\dagger \xrightarrow[N \otimes_B P \otimes \rho^l]{N \otimes \rho^r \otimes P^\dagger} N \otimes_B P \otimes_A P^\dagger \otimes_B P \otimes_A P^\dagger$$

is exact. Since  $P \in {}_B\mathcal{M}$  is faithfully flat, it suffices to show that  $S \otimes_B P$  is exact. It is clear that the sequence is a complex.

We first show that  $N \otimes_B \eta \otimes_B P$  is injective: if

$$0 = (N \otimes_B \eta \otimes_B P) \left( \sum_j n_j b_j \otimes_B p_j \right) = \sum_j n_j \otimes_B b_j^- \otimes_A b_j^+ \otimes_B p_j,$$

then

$$0 = \sum_j n_j \otimes_B b_j^- \varepsilon(b_j^+ \otimes_B p_j) = \sum_j n_j \otimes_B b_j p_j.$$

Now assume that

$$x = \sum_j n_j \otimes_B b_j p_j \otimes_A q_j \otimes_B c_j r_j \in \text{Ker} (N \otimes_B P \otimes_A \rho^l \otimes_B P - N \otimes_B \rho^r \otimes_A P^\dagger \otimes_B P).$$

Then

$$\sum_j n_j \otimes_B b_j p_j \otimes_A q_j \otimes_B c_j^- \otimes_A c_j^+ \otimes_B r_j = \sum_j n_j \otimes_B b_j^- \otimes_A b_j^+ \otimes_B p_j \otimes_A q_j c_j \otimes_B r_j,$$

and it follows that

$$\begin{aligned} x &= \sum_j n_j \otimes_B b_j p_j \otimes_A q_j \otimes_B c_j^- \varepsilon(c_j^+ \otimes_B r_j) \\ &= \sum_j n_j \otimes_B b_j^- \otimes_A b_j^+ \otimes_B p_j \varepsilon(q_j c_j \otimes_B r_j) \\ &= (N \otimes_B \eta \otimes_B P) \left( \sum_j n_j b_j \otimes_B p_j \varepsilon(q_j c_j \otimes_B r_j) \right). \end{aligned}$$

□

## 2. CORINGS FROM COLIMITS

Let  $F : \mathcal{Z} \rightarrow \mathcal{M}$  be a covariant functor. Recall (see for example [3]) that a *cocone* on  $F$  is a couple  $(M, m)$  where  $M \in \mathcal{M}$  and  $m_Z : F(Z) \rightarrow M$  is a morphism in  $\mathcal{M}$ , for every  $Z \in \mathcal{Z}$ , such that

$$(4) \quad m_Z = m_{Z'} \circ F(f),$$

for every  $f : Z \rightarrow Z'$  in  $\mathcal{Z}$ . The *colimit* of  $F$  is a cocone  $(C, c)$  on  $F$  satisfying the following universal property: if  $(M, m)$  is a cocone on  $F$ , then there exists a unique morphism  $f : C \rightarrow M$  in  $\mathcal{M}$  such that

$$(5) \quad f \circ c_Z = m_Z,$$

for every  $Z \in \mathcal{Z}$ . If the colimit exists, then it is unique up to isomorphism. We then write  $\text{colim } F = \text{colim } F(Z) = (C, c)$ .

The colimit  $(C, c)$  has the following property: if  $f, g : C \rightarrow M$  are two morphisms in  $\mathcal{M}$  such that  $f \circ c_Z = g \circ c_Z$ , for all  $Z \in \mathcal{Z}$ , then  $f = g$ . Indeed,  $(M, f \circ c = g \circ c)$  is a cocone on  $F$ , and  $f = g$  follows from the uniqueness in the definition of colimit.

From now on, let  $\mathcal{Z}$  be a (small) category and let  $(\mathcal{M}, \otimes, A)$  be a monoidal category. Then  $(\text{Func}(\mathcal{Z}, \mathcal{M}), \otimes, A)$  is also a monoidal category. The tensor  $\otimes$  and the unit  $A$  are given by the following formulas:

$$(F \otimes G)(Z) = F(Z) \otimes G(Z) \text{ and } (F \otimes G)(f) = F(f) \otimes G(f);$$

$$A(Z) = A \text{ and } A(f) = A,$$

for all  $F, G : \mathcal{Z} \rightarrow \mathcal{M}$ ,  $Z, Z' \in \mathcal{Z}$  and  $f : Z \rightarrow Z'$  in  $\mathcal{Z}$ . A coalgebra in  $(\text{Func}(\mathcal{Z}, \mathcal{M}), \otimes, A)$  will be called a  $\mathcal{Z}$ -coalgebra in  $\mathcal{M}$ . The result of this Section is the following.

**Proposition 2.1.** *Let  $(G, \Delta, \varepsilon)$  be a  $\mathcal{Z}$ -coalgebra in  $\mathcal{M}$ , and assume that  $\text{colim } G = (C, c)$  exists. Then  $C$  is a coalgebra in  $\mathcal{M}$ .*

*Proof.* We give a proof of the statement in case of a strict monoidal category  $\mathcal{M}$ . Recall that this is no restriction since every monoidal category is equivalent to a strict monoidal category, see for example [13, Prop. IX.5.1].

For every  $Z \in \mathcal{Z}$ , consider the morphism

$$d_Z = (c_Z \otimes c_Z) \circ \Delta_Z : G(Z) \rightarrow C \otimes C.$$

Let  $f : Z \rightarrow Z'$  be a morphism in  $\mathcal{Z}$ , and look at the diagram

$$\begin{array}{ccccc} G(Z) & \xrightarrow{\Delta_Z} & G(Z) \otimes G(Z) & \xrightarrow{c_Z \otimes c_Z} & C \otimes C \\ \downarrow G(f) & & \downarrow G(f) \otimes G(f) & & \downarrow = \\ G(Z') & \xrightarrow{\Delta_{Z'}} & G(Z') \otimes G(Z') & \xrightarrow{c_{Z'} \otimes c_{Z'}} & C \otimes C \end{array}$$

The left hand square commutes since  $\Delta : G \rightarrow G \otimes G$  is a natural transformation, and the right hand square commutes because  $(C, c)$  is a cocone on  $G$ . It follows that  $(C \otimes C, d)$  is a cocone on  $G$ , and we conclude that there exists a morphism  $\Delta_C : C \rightarrow C \otimes C$  in  $\mathcal{M}$  such that

$$\Delta_C \circ c_Z = d_Z = (c_Z \otimes c_Z) \circ \Delta_Z,$$

for all  $Z \in \mathcal{Z}$ . We then have

$$\begin{aligned} (\Delta_C \otimes C) \circ \Delta_C \circ c_Z &= (\Delta_C \otimes C) \circ (c_Z \otimes c_Z) \circ \Delta_Z \\ &= (c_Z \otimes c_Z \otimes C) \circ (\Delta_Z \otimes c_Z) \circ \Delta_Z \\ &= (c_Z \otimes c_Z \otimes c_Z) \circ (\Delta_Z \otimes G(Z)) \circ \Delta_Z \\ &= (c_Z \otimes c_Z \otimes c_Z) \circ (G(Z) \otimes \Delta_Z) \circ \Delta_Z \\ &= (C \otimes \Delta_C) \circ \Delta_C \circ c_Z, \end{aligned}$$

for all  $Z \in \mathcal{Z}$ . It follows (see [3, Prop. 2.6.4]) that  $(\Delta_C \otimes C) \circ \Delta_C = (C \otimes \Delta_C) \circ \Delta_C$ , so  $\Delta_C$  is a coassociative comultiplication on  $C$ .

The counit is defined in a similar way:  $(A, \varepsilon)$  is a cocone on  $G$ , so there exists a morphism  $\varepsilon_C : C \rightarrow A$  in  $\mathcal{M}$  such that  $\varepsilon_C \circ c_Z = \varepsilon_Z$ , for all  $Z \in \mathcal{Z}$ . The counit property is verified as follows: for all  $Z \in \mathcal{Z}$ , we have

$$(\varepsilon_C \otimes C) \circ \Delta_C \circ c_Z = (\varepsilon_C \otimes C) \circ (c_Z \otimes c_Z) \circ \Delta_Z = (A \otimes c_Z) \circ (\varepsilon_Z \otimes G(Z)) \circ \Delta_Z = c_Z.$$

□

**Proposition 2.2.** *Let  $(G, \Delta, \varepsilon)$  be as in Proposition 2.1. If  $(H, \rho)$  is a right  $G$ -comodule, and  $\text{colim } H = (M, m)$  exists, then  $M$  is a right  $C$ -comodule.*

*Proof.* For every  $Z \in \mathcal{Z}$ , consider the composition

$$r_Z = (m_Z \otimes c_Z) \circ \rho_Z : H(Z) \rightarrow H(Z) \otimes G(Z) \rightarrow M \otimes C.$$

Arguments similar to the ones presented above show that  $(M \otimes C, r)$  is a cocone on  $H$ . It follows that there exists a morphism  $\rho_M : M \rightarrow M \otimes C$  such that  $\rho_M \circ m_Z = r_Z$ , for every  $Z \in \mathcal{Z}$ . Standard computations show that  $\rho_M$  is coassociative and satisfies the counit property.  $\square$

### 3. SPLIT DIRECT SYSTEMS

Recall that a partially ordered set  $(I, \leq)$  is called *directed* if every finite subset of  $I$  has an upper bound. To a partially ordered set  $(I, \leq)$ , we can associate a category  $\mathcal{Z}$ . The objects of  $\mathcal{Z}$  are the elements of  $I$ , and  $\text{Hom}_{\mathcal{Z}}(i, j) = \{a_{ji}\}$  is a singleton if  $i \leq j$  and empty otherwise.

Let  $\mathcal{A}$  be a category and  $\mathcal{Z}$  a category associated to a directed partially ordered set. A functor  $\underline{M} : \mathcal{Z} \rightarrow \mathcal{A}$  will be called a *direct system* with values in  $\mathcal{A}$ . To  $\mathcal{A}$ , we associate a new category  $\mathcal{A}^s$ . The objects of  $\mathcal{A}$  and  $\mathcal{A}^s$  are the same. A morphism  $M \rightarrow N$  in  $\mathcal{A}^s$  is a couple  $(\mu, \nu)$ , with  $\mu : M \rightarrow N$  and  $\nu : N \rightarrow M$  in  $\mathcal{A}$  such that  $\nu \circ \mu = M$ , that is,  $\nu$  is a left inverse of  $\mu$ . A functor  $\underline{M}^s : \mathcal{Z} \rightarrow \mathcal{A}^s$  will be called a *split direct system* with values in  $\mathcal{A}$ . We will adopt the following notation, for all  $i \leq j \in I$ :

$$\underline{M}^s(i) = M_i, \quad \underline{M}^s(a_{ji}) = (\mu_{ji}, \nu_{ij}).$$

Then  $\mu_{ji} : M_i \rightarrow M_j$ ,  $\nu_{ij} : M_j \rightarrow M_i$ , and

$$(6) \quad \nu_{ij} \circ \mu_{ji} = M_i.$$

Consider the forgetful functor  $F : \mathcal{A}^s \rightarrow \mathcal{A}$ ,  $F(M) = M$ ,  $F(\mu, \nu) = \mu$ . Then  $F \circ \underline{M}^s = \underline{M}$  is a direct system with values in  $\mathcal{A}$ . In Proposition 3.1, we will assume that  $\text{colim } \underline{M} = (M, \mu)$  exists. This means in particular that we have morphisms  $\mu_i : M_i \rightarrow M$  such that

$$(7) \quad \mu_i = \mu_j \circ \mu_{ji}.$$

**Proposition 3.1.** *Let  $\underline{M}^s : \mathcal{Z} \rightarrow \mathcal{A}^s$  be a split direct system, and assume that  $\text{colim } \underline{M} = (M, \mu)$  exists. Then there exist unique morphisms  $\nu_i : M \rightarrow M_i$  in  $\mathcal{A}$  such that*

$$(8) \quad \nu_i \circ \mu_i = M_i \text{ and } \nu_i = \nu_{ij} \circ \nu_j,$$

for all  $i \leq j$  in  $I$ .

*Proof.* The proof in the case where  $\mathcal{A} = \mathcal{M}_A$  can be found in [17]. In the general case, we argue as follows. For a fixed  $i \in I$ , we have a cocone  $(M_i, u^i)$  on  $\underline{M}$  defined as follows: for every  $k \in I$ ,  $u_k^i : M_k \rightarrow M_i$  is the composition

$$\nu_{il} \circ \mu_{lk} : M_k \rightarrow M_l \rightarrow M_i,$$

where  $l \geq i, k$ . We have to show that this definition is independent of the choice of  $l$ . Take  $j \geq i, k$ , and  $m \geq l, j$ . Then

$$\nu_{im} \circ \mu_{mk} = \nu_{il} \circ \nu_{lm} \circ \mu_{ml} \circ \mu_{lk} = \nu_{il} \circ M_l \circ \mu_{lk} = \nu_{il} \circ \mu_{lk},$$

and, in a similar way,

$$\nu_{im} \circ \mu_{mk} = \nu_{ij} \circ \mu_{jk}.$$

$(M_i, u^i)$  is a cocone on  $\underline{M}$ : take  $k \geq j$  in  $I$ , and  $l \geq i, k$ ; then

$$u_k^i \circ \mu_{kj} = \nu_{il} \circ \mu_{lk} \circ \mu_{kj} = \nu_{il} \circ \mu_{lj} = u_j^i.$$

From the universal property of the colimit, it follows that there exists a unique  $\nu_i : M \rightarrow M_i$  such that

$$(9) \quad u_j^i = \nu_i \circ \mu_j,$$

for all  $j \in I$ . In particular,

$$\nu_i \circ \mu_i = u_i^i = \nu_{il} \circ \mu_{li} = M_i.$$

We have to show that  $\nu_i = \nu_{ij} \circ \nu_j$  if  $i \leq j$ . To this end, it suffices to show that

$$\nu_i \circ \mu_k = \nu_{ij} \circ \nu_j \circ \mu_k,$$

for all  $k \in I$ . We take  $l \geq j, k$  and compute

$$\nu_i \circ \mu_k = u_k^i = \nu_{il} \circ \mu_{lk} = \nu_{ij} \circ \nu_{jl} \circ \mu_{lk} = \nu_{ij} \circ u_k^j = \nu_{ij} \circ \nu_j \circ \mu_k.$$

We finally prove the uniqueness. Assume that  $\nu'_i : M \rightarrow M_i$  satisfies (8). Let  $i, j \in I$ , and take  $k \geq i, j$ . Then

$$\nu'_i \circ \mu_j \stackrel{(7,8)}{=} \nu_{ik} \circ \nu'_k \circ \mu_k \circ \mu_{kj} \stackrel{(8)}{=} \nu_{ik} \circ M_k \circ \mu_{kj} = \nu_{ik} \circ \mu_{kj} = u_j^i$$

so the  $\nu'_i$  satisfy (9). By the uniqueness in the definition of colimit, it follows that  $\nu'_i = \nu_i$ , for all  $i \in I$ .  $\square$

#### 4. COLIMIT COMATRIX CORINGS

Let  $k$  be a commutative ring. We say that a  $k$ -algebra  $B$  has *idempotent local units* if there exists a set of idempotent elements  $\{e_i \mid i \in I\} \subset B$  such that for every finite subset  $F \subset B$ , there exists  $i \in I$  such that  $e_i b = b e_i = b$ , for all  $b \in F$ . If, moreover, the  $e_i$  can be chosen to be orthogonal, then we say that  $B$  has *orthogonal idempotent local units*.

We will denote by  $\mathcal{F}_k$  the category of firm  $k$ -algebras.

**Lemma 4.1.** *The following statements are equivalent.*

- (i)  $B$  is a ring with idempotent local units;
- (ii) There exists a split direct system  $\underline{B}^s : \mathcal{Z} \rightarrow \mathcal{F}_k^s$  such that  $B = \text{colim}(\underline{B}, \beta)$ , where  $\beta_{ji} = \underline{B}(a_{ji})$  and  $B_i$  is a ring with unit;
- (iii) There exists a direct system  $\underline{B} : \mathcal{Z} \rightarrow \mathcal{F}_k$  such that  $\text{colim} \underline{B} = B$ , where  $\beta_{ji} = \underline{B}(a_{ji})$  and  $B_i$  is a ring with unit.

*Proof.* The statement follows immediately from Lemma 2.10 and the remark after Corollary 3.6 from [17]. However, for sake of completeness, let us repeat a full proof using the notation we introduced in the previous section.

$(i) \Rightarrow (ii)$ . On the index set  $I$  of the idempotent local units, we define a partial ordering  $\leq$  as follows:  $i \leq j$  if and only if  $e_i e_j = e_j e_i = e_i$ . This partial ordering is directed: for all  $i, j \in I$ , there exists  $k \in I$  such that  $k \geq i, j$ . Indeed, by definition of a  $k$ -algebra with idempotent local units, for the two elements  $e_i$  and  $e_j$ , we can find an element  $e_k$  with  $k \in I$ , such that  $e_k$  is a local unit for both  $e_i$  and  $e_j$ , i.e.  $k \geq i, j$ . Then let  $B_i = e_i B e_i$ , for each  $i \in I$ . If  $i \leq j$ , then  $B_i$  is a subalgebra of  $B_j$ , and the inclusion map  $\beta_{ji} : B_i \rightarrow B_j$  is a morphism in  $\mathcal{F}_k$ . Also  $\gamma_{ij} : B_j \rightarrow B_i$ ,  $\gamma_{ij}(b_j) = e_i b_j e_i$  is a morphism of firm algebras. Associate to the partially ordered directed set  $(I, \leq)$ , a category  $\mathcal{Z}$  as in Section 3, then we have a split direct system



$\underline{B}^s : \mathcal{Z} \rightarrow \mathcal{F}_k^s$ ,  $\underline{B}^s(i) = B_i$ ,  $\underline{B}^s(a_{ji}) = (\beta_{ji}, \gamma_{ij})$ . Clearly  $\text{colim } \underline{B} = (B, \beta)$ , with  $\beta_i : B_i \rightarrow B$  the inclusion map.

$(ii) \Rightarrow (iii)$  is trivial.

$(iii) \Rightarrow (i)$ . Recall that module categories contain colimits and they can be described as follows. Let

$$\mathcal{B} = \{(i, b_i) \mid i \in I, b_i \in B_i\}.$$

be the disjoint union of the  $B_i$ . An equivalence relation  $\sim$  on  $\mathcal{B}$  is defined as follows:  $(i, b_i) \sim (j, b_j)$  if and only if there exists  $k \geq i, j$  such that  $\beta_{ki}(b_i) = \beta_{kj}(b_j)$ , where  $\beta_{ki} = B(a_{ij})$ . Then let  $B = \mathcal{B} / \sim$  and  $\beta_i : B_i \rightarrow B$ ,  $\beta_i(b_i) = [(i, b_i)]$ . The elements of the form  $[(i, 1_{B_i})]$  make up a set of idempotent local units.  $\square$

*Remark 4.2.* Abrams [1] proved the implication (i) $\Rightarrow$ (iii) under the stronger assumption that  $B$  is a ring with *commuting* idempotent local units. Lemma 4.1 tells us that the implication still holds if we drop the condition that the idempotents commute, and then we even have an equivalence. Abrams [1, Lemma 1.5] also shows that firm modules over a ring with commuting idempotent local units can be written as direct limits. In Lemma 4.4, this property is generalized to arbitrary rings with idempotent local units, and it is shown that are precisely the ones that can be written as direct limits.

Let  $B$  be a  $k$ -algebra with idempotent local units and  $A$  a  $k$ -algebra with unit. Let  $I$  be the index set of idempotent local units of  $B$  and  $\mathcal{Z}$  the associated category as in Lemma 4.1.

**Lemma 4.3.**  *$P$  is a firm  $(B, A)$ -bimodule if and only if we can describe  $P$  in the following way. There exists a split direct system  $\underline{P}^s : \mathcal{Z} \rightarrow \mathcal{M}_A^s$  where we denote for all  $i \leq j \in I$ :*

$$\underline{P}^s(i) = P_i, \quad \underline{P}^s(a_{ji}) = (\sigma_{ji}, \tau_{ij}),$$

*and such that the following conditions hold*

- *for all  $i \leq j \in I$ ,  $b_i \in B_i$ ,  $p_j \in P_j$ :*

$$(10) \quad \beta_{ji}(b_i)p_j = \sigma_{ji}(b_i\tau_{ij}(p_j));$$

- *each  $P_i$  is a  $(B_i, A)$ -bimodule for the unital  $k$ -subalgebra  $B_i \subset B$  and*

-  *$\text{colim } \underline{P} = (P, \sigma)$ .*

*Proof.* This is an immediate consequence of [17, Lemmas 2.7 and 2.10]. For the sake of completeness, we give a complete proof in our present notation.

Suppose first that  $P$  is a firm  $(B, A)$  module. For each  $i \in I$ , we consider  $P_i = e_i P$ . Then  $P = \cup_{i \in I} P_i$ . Moreover it is clear that  $P_i$  is a left  $B_i = e_i B e_i$ -module and a  $(B_i, A)$ -bimodule. For  $i \leq j \in I$ , we have right  $A$ -module maps  $\sigma_{ji} : P_i \rightarrow P_j$  (the inclusion map) and  $\tau_{ij} : P_j \rightarrow P_i$ ,  $\tau_{ij}(p e_j) = p e_i$ . This defines a split direct system  $\underline{P}^s : \mathcal{Z} \rightarrow \mathcal{M}_A^s$ , and  $\text{colim } \underline{P} = (P, \sigma)$ , with  $\sigma_i : P_i \rightarrow P$  the inclusion map. Finally, we check that (10) holds in this situation. Let  $i \leq j$ , and take  $b_i = e_i b e_i \in B_i$ ,  $p_j = e_j p \in P_j$ . Then

$$\sigma_{ji}(b_i \tau_{ij}(p_j)) = e_i b e_i e_i e_j p = e_i b e_j p = b_i e_j p = \beta_{ji}(b_i) p_j,$$

as needed.

For the converse, the construction of the colimit is done using arguments similar to the ones in the proof of Lemma 4.1. From Proposition 3.1, we know that  $\sigma_i : P_i \rightarrow P$  has a left inverse  $\tau_i : P \rightarrow P_i$ . Take  $p \in P$ ,  $b \in B$ . Making use of the

characterisation of  $B$  given in Lemma 4.1, we can find  $i, j \in I$  such that  $p = \sigma_i(p_i)$ ,  $b = \beta_j(b_j)$ , with  $p_i \in P_i$ ,  $b_j \in B_j$ . Take  $k \geq i, j$ , and define

$$(11) \quad bp = \sigma_k(\beta_{kj}(b_j)\sigma_{ki}(p_i)).$$

To prove that this is a well-defined action of  $B$  on  $P$ , we have to show that (11) is independent of the choice of the index  $k$ . Suppose  $l \geq i, j$  and consider  $\sigma_l(\beta_{lj}(b_j)\sigma_{li}(p_i))$ . Take any  $m \geq l, k$  then we compute

$$\begin{aligned} \sigma_l(\beta_{lj}(b_j)\sigma_{li}(p_i)) &= \sigma_m \circ \sigma_{ml}(\beta_{lj}(b_j)\tau_{lm}\sigma_{ml}\sigma_{li}(p_i)) \\ &\stackrel{(10)}{=} \sigma_m(\beta_{ml}\beta_{lj}(b_j)\sigma_{ml}\sigma_{li}(p_i)) \\ &= \sigma_m(\beta_{mj}(b_j)\sigma_{mi}(p_i)) \end{aligned}$$

In a similar way, we prove that we can replace  $k$  by  $m$  in (11) and by this the left  $B$  action of  $P$  is independent of the choice of the index  $k$ . Finally,  $P$  is firm as a left  $B$ -module: take  $p = \sigma_i(p_i) \in P$ ; then  $\beta_i(1_{B_i})p = p$ .  $\square$

**Lemma 4.4.** *If  $P$  satisfies the equivalent conditions of Lemma 4.3, the following formulas hold, for all  $i \leq j \in I$ ,  $p_i \in P_i$ ,  $p_j \in P_j$ ,  $\varphi_i \in P_i^*$ ,  $\varphi_j \in P_j^*$  and  $b_i \in B_i$ .*

$$(12) \quad (\sigma_{ji} \circ \tau_{ij})(p_j) = \beta_{ji}(1_{B_i})p_j;$$

$$(13) \quad \sigma_{ji}(b_i p_i) = \beta_{ji}(b_i)\sigma_{ji}(p_i);$$

$$(14) \quad \varphi_j \beta_{ji}(b_i) = ((\varphi_j \circ \sigma_{ji})b_i) \circ \tau_{ij} = \tau_{ij}^*(\sigma_{ji}^*(\varphi_j)b_i);$$

$$(15) \quad \varphi_j \beta_{ji}(1_{B_i}) = \varphi_j \circ \sigma_{ji} \circ \tau_{ij} = \tau_{ij}^*(\sigma_{ji}^*(\varphi_j));$$

$$(16) \quad \tau_{ij}(\beta_{ji}(b_i)p_j) = b_i \tau_{ij}(p_j);$$

$$(17) \quad \tau_{ij}^*(\varphi_i b_i) = \tau_{ij}^*(\varphi_i)\beta_{ji}(b_i).$$

*Proof.* (12) follows after we take  $b_i = 1_{B_i}$  in (10). (13) can be shown as follows:

$$\beta_{ji}(b_i)\sigma_{ji}(p_i) \stackrel{(10)}{=} \sigma_{ji}(b_i(\tau_{ij} \circ \sigma_{ji})(p_i)) = \sigma_{ji}(b_i p_i).$$

We next prove (14). Take any  $p_j \in P_j$ ,

$$(\varphi_j \beta_{ji}(b_i))(p_j) = \varphi_j(\beta_{ji}(b_i)p_j) \stackrel{(10)}{=} (((\varphi_j \circ \sigma_{ji})b_i) \circ \tau_{ij})(p_j)$$

Then (15) follows after we take  $b_i = 1_{B_i}$  in (14), and (16) also follows easily:

$$\tau_{ij}(\beta_{ji}(b_i)p_j) \stackrel{(14)}{=} (\tau_{ij} \circ \sigma_{ji})(b_i \tau_{ij}(p_j)) = b_i \tau_{ij}(p_j).$$

(17) follows immediately from (16).  $\square$

**Lemma 4.5.** *If  $P$  satisfies the equivalent conditions of Lemma 4.3, we have a split direct system*

$$\underline{P}^{*s} : \mathcal{Z} \rightarrow {}_A\mathcal{M}^s, \quad \underline{P}^{*s}(i) = P_i^* = \text{Hom}_A(P_i, A), \quad \underline{P}^{*s}(a_{ji}) = (\tau_{ij}^*, \sigma_{ji}^*),$$

where for  $j \geq i$ , we have defined the maps

$$\sigma_{ji}^* : P_j^* \rightarrow P_i^*, \quad \sigma_{ji}^*(\varphi_j) = \varphi_j \circ \sigma_{ji},$$

$$\tau_{ij}^* : P_i^* \rightarrow P_j^*, \quad \tau_{ij}^*(\varphi_i) = \varphi_i \circ \tau_{ij}.$$

Furthermore  $\text{colim } \underline{P}^* = (P^\dagger, \tau^\dagger)$  exists and  $P^\dagger$  is a firm  $(A, B)$ -module.

*Proof.* It is straightforward to check that  $P^{*s}$  is a split direct system. Since  $P_i$  is a unital  $(B_i, A)$ -bimodule,  $P_i^*$  is a unital  $(A, B_i)$ -bimodule. The statement follows by Lemma 4.3 using left-right duality.  $\square$

We will now describe the colimit of  $P^{*s}$ .

**Lemma 4.6.** *Let  $i \in I$  and  $\varphi \in P^* = \text{Hom}_A(P, A)$ . There exists  $\varphi_i \in P_i^*$  such that*

$$\varphi = \varphi_i \circ \tau_i$$

*if and only if*

$$\varphi = \varphi \circ \sigma_i \circ \tau_i.$$

*In this situation,  $\varphi_i$  is unique, and is given by the formula  $\varphi_i = \varphi \circ \sigma_i$ ; furthermore, for every  $j \geq i$ ,  $\varphi = \varphi_j \circ \tau_j$ , with  $\varphi_j = \varphi_i \circ \tau_{ij}$ .*

*Proof.* If  $\varphi = \varphi_i \circ \tau_i$ , then  $\varphi \circ \sigma_i \circ \tau_i = \varphi_i \circ \tau_i \circ \sigma_i \circ \tau_i = \varphi_i \circ \tau_i = \varphi$ . The converse is obvious. If  $\varphi = \psi \circ \tau_i$ , then  $\varphi \circ \sigma_i = \psi \circ \tau_i \circ \sigma_i = \psi$ . If  $j \geq i$ , then  $\varphi_i \circ \tau_{ij} \circ \tau_j = \varphi_i \circ \tau_i = \varphi$ .  $\square$

Let  $P^\dagger = \{\varphi \in P^* \mid \exists i \in I : \varphi = \varphi \circ \sigma_i \circ \tau_i\}$ . More explicit, using the characterisations Lemma 4.1 and Lemma 4.3 we get  $P^\dagger = \{\varphi \in P^* \mid \exists i \in I : \varphi(p) = \varphi(e_i p), \text{ for all } p \in P\}$ . For every  $i \in I$ , we have a map

$$\tau_i^* : P_i^* \rightarrow P^\dagger, \tau_i^*(\varphi_i) = \varphi_i \circ \tau_i.$$

**Proposition 4.7.** *With notation as above,  $\text{colim } \underline{P}^* = (P^\dagger, \tau^*)$ .*

*Proof.* First,  $(P^\dagger, \tau^*)$  is a cocone on  $Q^*$  since, for all  $i \leq j$  and  $\varphi_i \in P_i^*$ , we have

$$(\tau_j^* \circ \tau_{ij}^*)(\varphi_i) = \varphi_i \circ \tau_{ij} \circ \tau_j = \varphi \circ \tau_i = \tau_i^*.$$

Let  $(M, m)$  be another cocone on  $Q^*$ . This means that  $m_i : P_i^* \rightarrow M$  and  $m_j \circ \tau_{ij}^* = m_i$  if  $i \leq j$ . We then define  $f : P^\dagger \rightarrow M$  as follows:  $f(\varphi_i \circ \tau_i) = m_i(\varphi_i)$ , for every  $i \in I$  and  $\varphi_i \in P_i^*$ . Let us show that  $f$  is well-defined. Assume that

$$\varphi = \varphi_i \circ \tau_i = \varphi_j \circ \tau_j.$$

Take  $k \geq i, j$ . Then  $\varphi = \varphi_k \circ \tau_k$  with  $\varphi_k = \varphi_i \circ \tau_{ik}$  (see Lemma 4.6). Then

$$m_k(\varphi_k) = m_k(\varphi_i \circ \tau_{ik}) = (m_k \circ \tau_{ik}^*)(\varphi_i) = m_i(\varphi_i).$$

In a similar way, we have that  $m_j(\varphi_j) = m_k(\varphi_k)$ , and it follows that  $f$  is well-defined. Finally,  $(f \circ \tau_i^*)(\varphi_i) = f(\varphi_i \circ \tau_i) = m_i(\varphi_i)$ .  $\square$

The right  $B$ -action on  $P^\dagger$  can be described as follows: take  $\varphi = \varphi_i \circ \tau_i \in P^\dagger$  and  $b = \beta_j(b_j) \in B$ . For  $k \geq i, j$ , we have

$$(18) \quad \varphi b = ((\varphi_i \circ \tau_{ik})\beta_{kj}(b_j)) \circ \tau_k.$$

In particular, we have, for  $\varphi_i \in P_i^*$  and  $b_i \in B_i$ :

$$(19) \quad (\varphi_i \circ \tau_i)\beta_i(b_i) = (\varphi_i b_i) \circ \tau_i.$$

In explicit form this means  $(\varphi b)(p) = \varphi_i(e_i e_j b_j p)$  or just  $(\varphi b)(p) = \varphi(bp)$ .

**Lemma 4.8.** *If  $P$  satisfies the equivalent conditions of Lemma 4.3, then we have for all  $i \in I$ ,  $b_i \in B_i$ ,  $p \in P$  and  $\varphi \in P^\dagger$ ,*

$$(20) \quad \beta_i(b_i)p = \sigma_i(b_i \tau_i(p));$$

$$(21) \quad \varphi \beta_i(b_i) = (\varphi \circ \sigma_i)b_i \circ \tau_i.$$

*Proof.* By the characterisation of Lemma 4.1 and Lemma 4.3, we can write  $b_i = e_i b e_i$  and  $\tau_i(p) = e_i p$ , where  $e_i$  is an idempotent in  $B$ . Moreover the maps  $\beta_i$  and  $\sigma_i$  are injections. With this information in hand we easily find

$$\begin{aligned}\beta_i(b_i)p &= e_i b e_i p \\ &= \sigma_i(b_i \tau_i(p)) = e_i b e_i e_i p.\end{aligned}$$

The other equation follows by

$$\varphi \beta_i(b_i)(p) = \varphi(\beta_i(b_i)p) = \varphi(\sigma_i(b_i \tau_i(p))) = (\varphi \circ \sigma_i)b_i \circ \tau_i(p),$$

where we used (20) in the second equality.  $\square$

**Proposition 4.9.** *We have a directed system  $\underline{G}: \mathcal{Z} \rightarrow {}_A\mathcal{M}_A$ ,  $G(i) = P_i^* \otimes_{B_i} P_i$ , and*

$$\underline{G}(a_{ji}): P_i^* \otimes_{B_i} P_i \rightarrow P_j^* \otimes_{B_j} P_j, \quad \underline{G}(a_{ji})(\varphi_i \otimes_{B_i} p_i) = \varphi_i \circ \tau_{ij} \otimes_{B_j} \sigma_{ji}(p_i).$$

*Proof.* We first show that  $\underline{G}(a_{ji})$  is well-defined. For all  $\varphi_i \in P_i^*$ ,  $p_i \in P_i$  and  $b_i \in B_i$ , we have

$$\begin{aligned}\underline{G}(a_{ji})(\varphi_i \otimes_{B_i} b_i \cdot p_i) &= \varphi_i \circ \tau_{ij} \otimes_{B_j} \sigma_{ji}(b_i p_i) \\ &\stackrel{(13)}{=} \tau_{ij}^*(\varphi_i) \otimes_{B_j} \beta_{ji}(b_i) \sigma_{ji}(p_i) = \tau_{ij}^*(\varphi_i) \beta_{ji}(b_i) \otimes_{B_j} \sigma_{ji}(p_i) \\ &\stackrel{(17)}{=} \tau_{ij}^*(\varphi_i b_i) \otimes_{B_j} \sigma_{ji}(p_i) = \underline{G}(a_{ji})(\varphi_i b_i \otimes_{B_i} p_i).\end{aligned}$$

If  $i \leq j \leq k$ , then we have

$$\begin{aligned}(\underline{G}(a_{kj}) \circ \underline{G}(a_{ji}))(\varphi_i \otimes_{B_i} t_i \cdot p_i) &= \varphi_i \circ \tau_{ji} \circ \tau_{jk} \otimes_{B_k} (\sigma_{kj} \circ \sigma_{ji})(p_i) \\ &= \varphi_i \circ \tau_{ik} \otimes_{B_k} \sigma_{ki}(p_i) = \underline{G}(a_{ki})(\varphi_i \otimes_{B_i} p_i).\end{aligned}$$

$\square$

Let  $P$  be a module satisfying the equivalent conditions of Lemma 4.3. Suppose that  $P_i$  is finitely generated and projective as right  $A$ -module for all  $i \in I$ . Let  $E_i = \sum z_i \otimes_A z_i^*$  be a finite dual basis of  $P_i \in \mathcal{M}_A$ ; we omitted the summation index.  $E_i$  is the unique element of  $P_i \otimes_A P_i^*$  satisfying the formulas

$$(22) \quad p_i = \sum z_i z_i^*(p_i); \quad \varphi_i = \sum z_i^* \varphi_i(z_i),$$

for all  $p_i \in P_i$  and  $\varphi_i \in P_i^*$ . With these notation, we have the following lemma.

**Lemma 4.10.** *(i) For all  $b_i \in B_i$ ,*

$$(23) \quad \sum b_i z_i \otimes_A z_i^* = \sum z_i \otimes_A z_i^* b_i,$$

*(ii) If  $i \leq j$ , then*

$$(24) \quad E_i = \sum \tau_{ij}(z_j) \otimes_A z_j^* \circ \sigma_{ji} = e_i z_j \otimes_A z_j|_{P_i}.$$

*Proof.* (i). This follows from the fact that  $P_i$  is a  $(B_i, A)$ -bimodule.

(ii). We show that the right hand side of (24) satisfies (22). For all  $p_i \in P_i$ , we have

$$\sum \tau_{ij}(z_j)(z_j^* \circ \sigma_{ji})(p_i) = \sum \tau_{ij}(z_j z_j^*(\sigma_{ji}(p_i))) = \tau_{ij}(\sigma_{ji}(p_i)) = p_i.$$

$\square$

For the remaining part of this paper, we will concentrate on modules that are locally projective in the sense of Ánh and Márki [2] (strongly locally projective in the terminology of [17]). We will need a more restrictive characterisation than Lemma 4.3. Recall first the definition of a morphism  $\eta : B \rightarrow B'$  of rings with (idempotent) local units. This is a ringmorphism  $\eta$  satisfying the property that for every finite subset  $F' \subset B'$ , we can find an (idempotent) local unit  $e_i \in B$  such that  $\eta(e_i)$  is an (idempotent) local unit for all elements of  $F'$ .

**Lemma 4.11.** *The following statements are equivalent*

- (i)  $P$  satisfies the equivalent conditions of Lemma 4.3, in addition  $P_i$  is finitely generated and projective as right  $A$ -module for all  $i \in I$  and  $\text{colim } \underline{P}^* = (P^\dagger, \tau^\dagger)$ .
- (ii)  $S = P \otimes_A P^\dagger$  is a ring with idempotent local units,  $P$  is a firm left  $S$ -module,  $P^\dagger$  is a firm right  $S$ -module and there exists a unique morphism of rings with idempotent local units  $\eta : B \rightarrow P \otimes_A P^\dagger$  such that

$$(25) \quad \eta(\beta_i(b_i)) = \sum \sigma_i(b_i z_i) \otimes_A z_i^* \circ \tau_i = \sum \sigma_i(z_i) \otimes_A z_i^* b_i \circ \tau_i,$$

for all  $i \in I$ ,  $b_i \in B_i$ .

- (iii)  $P$  is strongly  $P^\dagger$ -locally projective as right  $A$ -module and  $P^\dagger$  is strongly  $P$ -locally projective as left  $A$ -module.  $P$  is a firm left  $B$ -module and  $P^\dagger$  is a firm right  $B$ -module.

*Proof.* (i)  $\Rightarrow$  (ii) By Lemma 4.5  $\underline{P}^{*s}$  is a split direct system and obviously  $P_i^*$  is finitely generated and projective as left  $A$ -module for every  $i \in I$ . The first part of statement (ii) follows now from [17, Corollary 3.6]. The second equality in (25) is an immediate consequence of (23). Let us show that  $\eta$  is well-defined. Take  $b \in B$ , and assume that  $b = \beta_i(b_i) = \beta_j(b_j)$ , for some  $i, j \in I$ ,  $b_i \in B_i$ ,  $b_j \in B_j$ . Take  $k \geq i, j$ , and let  $b_k = \beta_{ki}(b_i) = \beta_{kj}(b_j)$ . We compute

$$\begin{aligned} \sum \sigma_i(b_i z_i) \otimes_A z_i^* \circ \tau_i &\stackrel{(24)}{=} \sum (\sigma_k \circ \sigma_{ki})(b_i \tau_{ik}(z_k)) \otimes_A z_k^* \circ \sigma_{ki} \circ \tau_{ik} \circ \tau_k \\ &\stackrel{(10,16)}{=} \sum \sigma_k(\beta_{ki}(b_i) z_k) \otimes_A (z_k^* \beta_{ki}(1_{B_i})) \circ \tau_k \\ &\stackrel{(23)}{=} \sum \sigma_k(\beta_{ki}(1_{B_i}) \beta_{ki}(b_i) z_k) \otimes_A z_k^* \circ \tau_k \\ &= \sum \sigma_k(\beta_{ki}(1_{B_i} b_i) z_k) \otimes_A z_k^* \circ \tau_k \\ &= \sum \sigma_k(b_k z_k) \otimes_A z_k^* \circ \tau_k. \end{aligned}$$

In a similar way, we prove that

$$\sum \sigma_j(b_j z_j) \otimes_A z_j^* \circ \tau_j = \sum \sigma_k(b_k z_k) \otimes_A z_k^* \circ \tau_k,$$

and it follows that the right hand side of (25) is independent of the choice of  $i$ . Next we prove that  $\eta$  is a ringmorphism. Take two elements  $b, b' \in B$  and choose  $i$  big enough such that  $b = \beta_i(b_i)$  and  $b' = \beta_i(b'_i)$ . Let us denote  $E_i = \sum z_i \otimes_A z_i^* = \sum \tilde{z}_i \otimes_A \tilde{z}_i^*$

$$\begin{aligned} \eta(b)\eta(b') &= \sum \sigma_i(b_i z_i) z_i^* \circ \tau_i \circ \sigma_i(b'_i \tilde{z}_i) \otimes_A \tilde{z}_i^* \circ \tau_i \\ &= \sum \sigma_i(b_i b'_i \tilde{z}_i) \otimes_A \tilde{z}_i^* \circ \tau_i = \eta(bb') \end{aligned}$$

Finally, the idempotent local units of  $P \otimes_A P^\dagger$  are of the form  $\sum \sigma_i(z_i) \otimes_A z_i^* \circ \tau_i$ , these are exactly given by  $\eta(1_{B_i})$ , so  $\eta$  is a morphism of rings with idempotent local units.

(ii)  $\Rightarrow$  (iii). By [17, Corollary 3.6] we only have to prove that  $P$  and  $P^\dagger$  are firm  $B$ -modules under the action induced by the morphism  $\eta$ . This is a consequence of the fact that  $\eta$  is a morphism of rings with enough idempotents. Take  $p \in P$ , then we know there exists an idempotent  $e \in B$  such that  $\eta(e) \in P \otimes_A P^\dagger$  is a local unit for  $p$ . Thus  $e \cdot p = \eta(e)p = p$  and  $P$  is a firm  $B$ -module. Analogously one proves  $P^\dagger$  is a firm right  $B$ -module.

(iii)  $\Rightarrow$  (i). Follows from [17, Corollary 3.6] and Lemma 4.3.  $\square$

For every  $i \in I$ , consider bimodule maps

$$\begin{aligned} \text{coev}_{P_i} : B_i &\rightarrow P_i \otimes_A P_i^*, & \text{coev}_{P_i}(b_i) &= b_i E_i = E_i b_i \\ \text{ev}_{P_i} : P_i^* \otimes_{B_i} P_i, & & \text{ev}_{P_i}(\varphi_i \otimes_{B_i} p_i) &= \varphi_i(p_i) \end{aligned}$$

Then  $(B_i, A, P_i, P_i^*, \text{coev}_{P_i}, \text{ev}_{P_i})$  is a comatrix coring context, so we have a comatrix coring  $(\underline{G}(i), \Delta_i, \varepsilon_i)$  with

$$\Delta_i(\varphi_i \otimes_{B_i} p_i) = \varphi_i \otimes_{B_i} E_i \otimes_{B_i} p_i \text{ and } \varepsilon_i(\varphi_i \otimes_{B_i} p_i) = \varphi_i(p_i).$$

$\underline{G}(i)$  is a finite comatrix coring, as introduced in [10].

**Proposition 4.12.** *Suppose the equivalent conditions of Lemma 4.11 hold, and consider the directed system  $\underline{G}$  from Proposition 4.9. Then  $(\underline{G}, \Delta, \varepsilon)$  is a coalgebra in  $\text{Func}(\mathcal{Z}, {}_A\mathcal{M}_A)$ .*

*Proof.* It suffices to show that  $\Delta$  and  $\varepsilon$  are natural transformations, or, equivalently, that  $\underline{G}(a_{ji})$  is a morphism of corings, for every  $i \leq j$ , or

$$(\underline{G}(a_{ji}) \otimes_A \underline{G}(a_{ji})) \circ \Delta_i = \Delta_j \circ \underline{G}(a_{ji}) ; \varepsilon_i = \varepsilon_j \circ \underline{G}(a_{ji}).$$

For all  $\varphi_i \in P_i^*$  and  $p_i \in P_i$ , we compute

$$\begin{aligned} (\Delta_j \circ \underline{G}(a_{ji}))(\varphi_i \otimes_{B_i} p_i) &= \Delta_j(\varphi_i \circ \tau_{ij} \otimes_{B_j} \sigma_{ji}(p_i)) \\ &= \varphi_i \circ \tau_{ij} \otimes_{B_j} E_j \otimes_{B_j} \sigma_{ji}(p_i) \\ &= \sum \varphi_i \circ \tau_{ij} \circ \sigma_{ji} \circ \tau_{ij} \otimes_{B_j} z_j \otimes_A z_j^* \otimes_{B_j} (\sigma_{ji} \circ \tau_{ij} \circ \sigma_{ji})(p_i) \\ &\stackrel{(12,16)}{=} \sum \varphi_i \circ \tau_{ij} \otimes_{B_j} (\sigma_{ji} \circ \tau_{ij})(z_j) \otimes_A z_j^* \circ \sigma_{ji} \circ \tau_{ij} \otimes_{B_j} \sigma_{ji}(p_i) \\ &\stackrel{(24)}{=} \sum \varphi_i \circ \tau_{ij} \otimes_{B_j} \sigma_{ji}(z_i) \otimes_A z_i^* \circ \tau_{ij} \otimes_{B_j} \sigma_{ji}(p_i) \\ &= \sum \underline{G}(a_{ji})(\varphi_i \otimes_{B_i} z_i) \otimes_A \underline{G}(a_{ji})(z_i^* \otimes_{B_i} p_i) \\ &= ((\underline{G}(a_{ji}) \otimes_A \underline{G}(a_{ji})) \circ \Delta_i)(\varphi_i \otimes_{B_i} p_i) \end{aligned}$$

and

$$\varepsilon_j(\underline{G}(a_{ji})(\varphi_i \otimes_{B_i} p_i)) = (\varphi_i \circ \tau_{ij} \circ \sigma_{ji})(p_i) = \varphi_i(p_i) = \varepsilon_i(\varphi_i \otimes_{B_i} p_i).$$

$\square$

**Proposition 4.13.** *Under the same conditions as Proposition 4.12,  $\text{colim } \underline{G} = (P^\dagger \otimes_B P, g)$ , with*

$$g_i : \underline{G}(i) = P_i^* \otimes_{B_i} P_i \rightarrow P^\dagger \otimes_B P, \quad g_i(\varphi_i \otimes_{B_i} p_i) = \varphi_i \circ \tau_i \otimes_B \sigma_i(p_i).$$

*Proof.* We first show that  $g_i$  is well-defined. For all  $b_i \in B_i$ , we have

$$\begin{aligned} g_i(\varphi_i b_i \otimes p_i) &= \varphi_i b_i \circ \tau_i \otimes_B \sigma_i(p_i) = (\varphi_i \circ \tau_i \circ \sigma_i) b_i \circ \tau_i \otimes_B \sigma_i(p_i) \\ &\stackrel{(21)}{=} (\varphi_i \circ \tau_i) \beta_i(b_i) \otimes_B \sigma_i(p_i) = (\varphi_i \circ \tau_i) \otimes_B \beta_i(b_i) \sigma_i(p_i) \\ &\stackrel{(20)}{=} (\varphi_i \circ \tau_i) \otimes_B \sigma_i(b_i \tau_i(\sigma_i(p_i))) = (\varphi_i \circ \tau_i) \otimes_B \sigma_i(bp_i) = g_i(\varphi_i \otimes b_i p_i). \end{aligned}$$

Let us now prove that  $(P^\dagger \otimes_B P, g)$  is a cocone on  $\underline{G}$ . Indeed, if  $i \leq j$ , then

$$\begin{aligned} (g_j \circ \underline{G}(a_{ji}))(\varphi_i \otimes_{B_i} p_i) &= g_j(\varphi_i \circ \tau_{ij} \otimes_{B_j} \sigma_{ji}(p_i)) \\ &= \varphi_i \circ \tau_{ij} \circ \tau_j \otimes_B \sigma_j(\sigma_{ji}(p_i)) = \varphi_i \circ \tau_i \otimes_B \sigma_i(p_i) = g_i(\varphi_i \otimes_{B_i} p_i). \end{aligned}$$

Let  $(M, m)$  be another cocone on  $\underline{G}$ . Then  $m_i : P_i^* \otimes_{B_i} P_i \rightarrow M$  and  $m_j \circ \underline{G}(a_{ji}) = m_i$  if  $j \geq i$ . We define  $f : P^\dagger \otimes_B P \rightarrow M$  as follows. For  $\varphi \in P^\dagger$  and  $p \in P$ , we can find  $i \in I$ ,  $\varphi_i \in P_i^*$  and  $p_i \in P_i$  such that  $p = \sigma_i(p_i)$  and  $\varphi = \varphi_i \circ \tau_i$ ; we then define

$$f(\varphi \otimes p) = m_i(\varphi_i \otimes_{T_i} p_i).$$

We have to show that  $f$  is well-defined. If  $k \geq i$ , then we have that  $\varphi = \varphi_k \circ \tau_k$  and  $p = \sigma_k(p_k)$  with  $\varphi_k = \varphi_i \circ \tau_{ik}$  and  $p_k = \sigma_{ki}(p_i)$ . We then find that

$$\begin{aligned} m_k(\varphi_k \otimes_{B_k} p_k) &= m_k(\varphi_i \circ \tau_{ik} \otimes_{B_k} \sigma_{ki}(p_i)) \\ &= (m_k \circ \underline{G}(a_{ki}))(\varphi_i \otimes_{B_i} p_i) = m_i(\varphi_i \otimes_{B_i} p_i). \end{aligned}$$

We will now show that  $f$  induces a map  $f : P^\dagger \otimes_B P \rightarrow M$ . To this end, we need to prove that

$$f(\varphi b \otimes p) = f(\varphi \otimes bp),$$

for all  $\varphi \in P^\dagger$ ,  $p \in P$  and  $b \in B$ . We can find  $i \in I$ ,  $b_i \in B_i$ ,  $\varphi \in P_i^*$  and  $p_i \in P_i$  such that  $b = \beta_i(b_i)$ ,  $p = \sigma_i(p_i)$  and  $\varphi = \varphi_i \circ \tau_i$ . Then we compute that

$$\begin{aligned} f(\varphi b \otimes p) &= f((\varphi_i \circ \tau_i) \beta_i(b_i) \otimes \sigma_i(p_i)) \\ &\stackrel{(21)}{=} f(((\varphi_i \circ \tau_i \circ \sigma_i) b_i) \circ \tau_i \otimes \sigma_i(p_i)) = f(\varphi_i b_i \circ \tau_i \otimes \sigma_i(p_i)) \\ &= m_i(\varphi_i b_i \otimes_{B_i} p_i) = m_i(\varphi_i \otimes_{B_i} b_i p_i) \\ &\stackrel{(20)}{=} f(\varphi_i \circ \tau_i \otimes \sigma_i(b_i p_i)) = f(\varphi_i \circ \tau_i \otimes \sigma_i(b_i \tau_i(\sigma_i(p_i)))) \\ &\stackrel{(20)}{=} f(\varphi_i \circ \tau_i \otimes \beta_i(b_i) \sigma_i(\tau_i(\sigma_i(p_i)))) = f(\varphi \otimes bp). \end{aligned}$$

Finally,

$$f(g_i(\varphi_i \otimes_{B_i} p_i)) = f(\varphi_i \circ \tau_i \otimes_B \sigma_i(p_i)) = m_i(\varphi_i \otimes_{B_i} p_i).$$

□

The following result now follows immediately from Propositions 2.1, 4.12 and 4.13.

**Corollary 4.14.** *If the equivalent conditions of Lemma 4.11 hold,  $\mathcal{G} = P^\dagger \otimes_B P$  is an  $A$ -coring, with comultiplication and counit given by the following formulas, for all  $i \in I$ ,  $\varphi_i \in P_i^*$  and  $p_i \in P_i$ :*

$$\begin{aligned} \Delta(\varphi_i \circ \tau_i \otimes_B \sigma_i(p_i)) &= \sum \Delta(\varphi_i \circ \tau_i \otimes_B \sigma_i(z_i) \otimes_A z_i^* \circ \tau_i \otimes_B \sigma_i(p_i)), \\ \varepsilon(\varphi_i \circ \tau_i \otimes_B \sigma_i(p_i)) &= \varphi_i(p_i). \end{aligned}$$

As before,  $E_i = \sum z_i \otimes_A z_i^*$  is the finite dual basis of  $P_i \in \mathcal{M}_A$ .

We will now show that  $P^\dagger \otimes_B P$  can be constructed starting from a Gómez-Vercruysse comatrix coring context, as described in Section 1. We already know that  $P$  and  $P^\dagger$  are firm bimodules.

**Proposition 4.15.** *If the equivalent conditions of Lemma 4.11 hold,  $(B, A, P, P^\dagger, \eta, \varepsilon)$  is a comatrix coring context, where  $\varepsilon : P^\dagger \otimes_B P \rightarrow A$  is the restriction of the evaluation map  $P^* \otimes_B P \rightarrow A$ .*

*Proof.* We have to show that (2) holds. Take  $b = \beta_i(b_i) \in B$ ,  $p = \sigma_i(p_i) \in P$  and  $\varphi = \varphi_i \circ \tau_i \in P^\dagger$ . Then

$$\begin{aligned} b^- \varepsilon(b^+ \otimes_A p) &= \sum \sigma_i(b_i z_i)(z_i^* \circ \tau_i)(\sigma_i(p_i)) \\ &= \sum \sigma_i(b_i z_i z_i^*(p_i)) = \sigma_i(b_i p_i) = \beta_i(b_i) \sigma_i(p_i) = bp, \end{aligned}$$

and

$$\begin{aligned} (\varepsilon(\varphi \otimes_A b^-) b^+)(p) &= \left( \sum \varphi(\sigma_i(z_i))(z_i^* b_i) \circ \tau_i \right)(p) = \sum \varphi(\sigma_i(z_i)) z_i^*(b_i \tau_i(p)) \\ &= \sum \varphi(\sigma_i(z_i z_i^*(b_i p_i))) = \varphi(\sigma_i(b_i p_i)) = \varphi(bp) = (\varphi b)(p), \end{aligned}$$

hence  $\varepsilon(\varphi \otimes_A b^-) b^+ = \varphi b$ .  $\square$

**Example 4.16.** Let  $B$  be a  $k$ -algebra with orthogonal idempotent local units and let  $\{e_i \mid i \in I\}$  be a complete set of idempotents. For all  $i, j \in I$ , let  $B_{ij} = e_i B e_j$ . Then  $B = \bigoplus_{i,j \in I} B_{ij}$ , and a firm left  $B$ -module  $P$  can then be written as  $P = \bigoplus_{i \in I} P_i$ , with  $P_i = e_i P$  a left  $B_i = B_{ii}$ -module. For each  $i \in I$ , we take a  $(B_i, A)$ -bimodule  $P_i$  which is finitely generated and projective as a right  $A$ -module, and we put  $P = \bigoplus_{i \in I} P_i$ . It is not hard to see that  $P^\dagger = \bigoplus_{i \in I} P_i^*$ , and we have a comatrix coring  $P^\dagger \otimes_B P$ . This way we recover the comatrix corings that were considered first in [11, Proposition 5.2].

**Example 4.17.** As a special case of the previous example, consider now the case where the orthogonal idempotents are central in  $B$ , then the situation simplifies to  $B = \bigoplus_{i \in I} B_i$ , where  $B_i = B e_i$ .

The functor  $K : \mathcal{M}_B \rightarrow \mathcal{M}^{\mathcal{G}}$  can be described as follows. Take  $M \in \mathcal{M}_B$ , and, as in Lemma 4.3, let  $M_i = M e_i$ . We have a split direct system  $\underline{F} : \mathcal{Z} \rightarrow \mathcal{M}_A^s$ :

$$\underline{F}(i) = M_i \otimes_{B_i} P_i ; \quad \underline{F}(a_{ji}) = (\mu_{ji} \otimes \sigma_{ji}, \nu_{ij} \otimes \tau_{ij}).$$

Then  $K(M) = \text{colim } \underline{F}$ , with the obvious coaction.

In view of Theorem 1.3, it is important to know when  $P \in {}_B \mathcal{M}$  is (faithfully) flat. We have the following results.

**Proposition 4.18.** *Let  $B$  be a  $k$ -algebra with idempotent local units, and take  $P \in {}_B \mathcal{M}$ . If for every  $i \in I$ , there exists  $j \geq i$  such that  $P_j \in {}_{B_j} \mathcal{M}$  is flat, then  $P \in {}_B \mathcal{M}$  is flat.*

*Proof.* Let  $f : N' \rightarrow N$  be an injective map in  $\mathcal{M}_B$ , and  $x \in \ker(f \otimes_B P)$ .  $N' \otimes_B P$  is the colimit of the  $N'_i \otimes_{B_i} P_i$ , so  $x$  can be represented by  $\sum_r n'_r \otimes_{B_i} p_r$  with  $n'_r \in N'_i$ ,  $p_r \in P_i$ .  $\sum_r f(n'_r) \otimes_{B_i} p_r$  represents zero in  $N \otimes_B P$ , so, replacing  $i$  by a bigger index, we can assume that  $\sum_r f(n'_r) \otimes_{B_i} p_r = 0 \in N_i \otimes_{B_i} P_i$ . Replace  $i$  by a bigger index such that  $P_i \in {}_{B_i} \mathcal{M}$  is flat. Then  $\sum_r n'_r \otimes_{B_i} p_r = 0$  in  $N'_i \otimes_{B_i} P_i$ , and this implies that  $x = 0$ .  $\square$

**Proposition 4.19.** *Let  $B$  be a  $k$ -algebra with idempotent local units, and assume that  $P \in {}_B \mathcal{M}$  is (faithfully) flat. If  $i \in I$  is such that  $e_i$  is central in  $B$ , then  $P_i$  is (faithfully) flat as a left  $B_i$ -module.*



*Proof.* Take  $N \in \mathcal{M}_{B_i}$ . We have  $\gamma_i : B \rightarrow B_i$ , making  $N \in \mathcal{M}_B$  via restriction of scalars. Then we claim that we have an isomorphism of  $k$ -modules

$$(26) \quad N \otimes_B P \cong N \otimes_{B_i} P_i.$$

Indeed, the map

$$f : N \otimes_{B_i} P_i \rightarrow N \otimes_B P, \quad f(n \otimes_{B_i} p_i) = n \otimes_B p_i$$

has an inverse  $g$  given by

$$g(n \otimes_B p) = n \otimes_{B_i} e_i p.$$

$g$  is well-defined since

$$g(nb \otimes p) = g(ne_i b \otimes p) = g(ne_i b e_i \otimes p) = ne_i b e_i \otimes_{B_i} e_i p = n \otimes_{B_i} e_i b e_i p = g(n \otimes bp).$$

Assume that  $P \in {}_B\mathcal{M}$  is faithfully flat. A sequence

$$0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$$

is exact in  $\mathcal{M}_{B_i}$  if and only if

$$0 \rightarrow N' \otimes_B P \rightarrow N \otimes_B P \rightarrow N'' \otimes_B P \rightarrow 0$$

is exact in  $\mathcal{M}_k$ , and, by (26), this is equivalent to exactness of the sequence

$$0 \rightarrow N' \otimes_{B_i} P_i \rightarrow N \otimes_{B_i} P_i \rightarrow N'' \otimes_{B_i} P_i \rightarrow 0.$$

□

We remark that the condition that  $e_i$  is central is fulfilled in the situation of Example 4.17. The condition that the  $e_i$  are central is also needed in the proof of our next result. We have seen that the comatrix coring is the colimit of the directed system  $\underline{G}$  discussed in Proposition 4.9. If we work over an algebra with central idempotent local units, then this system is split.

**Proposition 4.20.** *Let  $B$  be a  $k$ -algebra with central idempotent local units and suppose the equivalent conditions of Lemma 4.11 hold, then the direct system  $\underline{G}$  of Proposition 4.9 splits.  $\underline{G}^s : \mathcal{Z} \rightarrow {}_A\mathcal{M}_A^s$ , with  $\underline{G}^s(a_{ji}) = (g_{ji}, h_{ij})$ , where*

$$h_{ij}(\varphi_j \otimes_{B_j} p_j) = \varphi_j \circ \sigma_{ji} \otimes_{B_i} \tau_{ij}(p_j) = \varphi_j|_{P_i} \otimes_{B_i} e_i p_j,$$

for all  $\varphi_j \in P_j^*$  and  $p_j \in P_j$ .

*Proof.* Let us show that  $h_{ij}$  is well-defined; all the rest is obvious. First we compute for  $\varphi_j \in P_j^*$ ,  $b_j \in B_j$  and  $p_i \in P_i$  that

$$(\varphi_j b_j)(\sigma_{ji} p_i) = \varphi_j(b_j p_i) = \varphi_j(b_j e_i p_i) = \varphi_j(e_i b_j e_i p_i) = (\varphi_j \circ \sigma_{ji})(e_i b_j e_i)(p_i),$$

where we used the fact that  $e_i$  is central. Then we compute

$$\begin{aligned} h_{ij}(\varphi_j b_j \otimes p_j) &= \varphi_j b_j \circ \sigma_{ji} \otimes_{B_i} e_i p_j = (\varphi_j \circ \sigma_{ji})(e_i b_j e_i) \otimes_{B_i} e_i p_j \\ &= \varphi_j \circ \sigma_{ji} \otimes_{B_i} e_i b_j e_i p_j = \varphi_j \circ \sigma_{ji} \otimes_{B_i} e_i b_j p_j = h_{ij}(\varphi_j \otimes b_j p_j). \end{aligned}$$

□

## 5. FACTORIZING SPLIT DIRECT SYSTEMS

In this Section, we consider split direct systems  $\underline{P}^s : \mathcal{Z} \rightarrow \mathcal{M}_{\mathcal{A}, \text{fgp}}^s$  that factorize through a  $k$ -linear category  $\mathcal{A}$ : we assume that there exists a split direct system

$$\underline{M}^s : \mathcal{Z} \rightarrow \mathcal{A}^s, \quad \underline{M}^s(i) = M_i, \quad \underline{M}^s(a_{ji}) = (\mu_{ji}, \nu_{ij})$$

and a functor  $\omega : \mathcal{A} \rightarrow \mathcal{M}_{\mathcal{A}}$  such that  $\underline{P}^s = \omega \circ \underline{M}^s$ , or

$$P_i = \omega(M_i), \quad \sigma_{ji} = \omega(\mu_{ji}), \quad \tau_{ij} = \omega(\nu_{ij}).$$

For every  $i \in I$ ,  $T_i = \text{End}_{\mathcal{A}}(M_i)$  is a  $k$ -algebra with unit. For  $i \leq j$ , we have a multiplicative map

$$\rho_{ji} : T_i \rightarrow T_j, \quad \rho_{ji}(t_i) = \mu_{ji} \circ t_i \circ \nu_{ij}.$$

This defines a direct system  $\underline{T} : \mathcal{Z} \rightarrow \mathcal{F}_k$ ,  $\underline{T}_i = T_i$ ,  $\underline{T}(a_{ji}) = \rho_{ji}$ . If  $t_i \in T_i = \text{End}_{\mathcal{A}}(M_i)$ , then  $\omega(t_i) \in \text{End}_{\mathcal{A}}(P_i)$ . Hence  $P_i$  is a  $(T_i, \mathcal{A})$ -bimodule, with left  $T_i$ -action given by

$$t_i \cdot p_i = \omega(t_i)(p_i).$$

We claim that (10) holds. Indeed, for all  $i \leq j$ ,  $t_i \in T_i$  and  $p_j \in P_j$ , we have

$$\begin{aligned} \rho_{ji}(t_i) \cdot p_j &= (\mu_{ji} \circ t_i \circ \nu_{ij}) \cdot p_j = \omega(\mu_{ji} \circ t_i \circ \nu_{ij})(p_j) \\ &= (\sigma_{ji} \circ \omega(t_i) \circ \tau_{ij})(p_j) = \sigma_{ji}(t_i \cdot \tau_{ij}(p_j)). \end{aligned}$$

Applying the results of Section 4, we obtain a comatrix coring. We will now assume that  $\text{colim } \underline{M} = (M, \mu)$  exists, and that  $\omega$  preserves colimits. We will give an explicit description of  $\text{colim } \underline{T}$ , and provide some alternative descriptions of the comatrix coring. Using Proposition 3.1, we obtain morphisms  $\nu_i : M \rightarrow M_i$ . Let  $\sigma_i = \omega(\mu_i)$ ,  $\tau_i = \omega(\nu_i)$ . We consider the  $k$ -algebra  $T = \text{End}_{\mathcal{A}}(M)$ . For every  $i \in I$ ,  $e_i = \mu_i \circ \nu_i$  is an idempotent in  $T$ . We also have

$$(27) \quad e_i \circ \mu_i = \mu_i \text{ and } \nu_i \circ e_i = \nu_i,$$

and, for  $i \leq j$ :

$$(28) \quad e_j \circ e_i = e_i \circ e_j = e_i.$$

(27) is immediate; (28) can be seen as follows:

$$\begin{aligned} e_j \circ e_i &\stackrel{(27)}{=} e_j \circ \mu_i \circ \nu_i = e_j \circ \mu_j \circ \mu_{ji} \circ \nu_i \\ &\stackrel{(27)}{=} \mu_j \circ \mu_{ji} \circ \nu_i = \mu_i \circ \nu_i = e_i; \\ e_i \circ e_j &\stackrel{(27)}{=} \mu_i \circ \nu_i \circ e_j = \mu_i \circ \nu_{ij} \circ \nu_j \circ e_j \\ &\stackrel{(27)}{=} \mu_i \circ \nu_{ij} \circ \nu_j = \mu_i \circ \nu_i = e_i. \end{aligned}$$

**Lemma 5.1.** *Let  $i \in I$  and  $t \in T$ . There exists  $t_i \in T_i$  such that*

$$t = \mu_i \circ t_i \circ \nu_i$$

*if and only if*

$$t = e_i \circ t \circ e_i.$$

*In this situation,  $t_i$  is unique, and is given by the formula  $t_i = \nu_i \circ t \circ \mu_i$ ; furthermore, for every  $j \geq i$ ,  $t = \mu_j \circ t_j \circ \nu_j$ , with*

$$(29) \quad t_j = \mu_{ji} \circ t_i \circ \nu_{ij}.$$

*Proof.* We leave the first part as an easy exercise to the reader. For  $j \geq i$ , we compute

$$\mu_j \circ \mu_{ji} \circ t_i \circ \nu_{ij} \circ \nu_j = \mu_i \circ t_i \circ \nu_i = t.$$

□

**Proposition 5.2.**  $T^\dagger = \{t \in T \mid \exists i \in I : t = e_i \circ t \circ e_i\}$  is a subalgebra of  $T$  with idempotent local units. In particular,  $T^\dagger$  is a firm  $k$ -algebra.  $\text{colim } \underline{T} = (T^\dagger, \rho)$ , with

$$\rho_i : T_i \rightarrow T^\dagger, \rho_i(t_i) = \mu_i \circ t_i \circ \nu_i.$$

*Proof.* It is clear that the  $e_i$  form a set of idempotent local units. It follows from Lemma 5.1 that  $T_i = e_i T e_i$ .  $(T^\dagger, \rho)$  is a cocone on  $T$  since, for all  $j \geq i$  and  $t_i \in T_i$ , we have

$$(\rho_j \circ \rho_{ji})(t_i) = \mu_j \circ \mu_{ji} \circ t_i \circ \nu_{ij} \circ \nu_j = \mu_i \circ t_i \circ \nu_i = \rho_i(t_i).$$

Assume that  $(M, m)$  is another cocone on  $T^\dagger$ . This means that  $m_i : T_i \rightarrow M$  and  $m_j \circ \rho_{ji} = m_i$  if  $i \leq j$ . The map  $f : T^\dagger \rightarrow M$ ,  $f(\mu_i \circ t_i \circ \nu_i) = m_i(t_i)$ , is well-defined: assume that  $t = \mu_i \circ t_i \circ \nu_i = \mu_j \circ t_j \circ \nu_j$ . Take  $k \geq i, j$ . By (27),  $t = \mu_k \circ t_k \circ \nu_k$ , with  $t_k = \mu_{ki} \circ t_i \circ \nu_{ik} = \rho_{ki}(t_i)$ , and it follows that  $m_k(t_k) = m_k(\rho_{ki}(t_i)) = m_i(t_i)$ . In a similar way, we have that  $m_k(t_k) = m_j(t_j)$ .

Finally, for every  $i \in I$  and  $t_i \in T_i$ , we have that  $(f \circ \rho_i)(t_i) = f(\mu_i \circ t_i \circ \nu_i) = m_i(t_i)$ . □

It is easy to show that  $P_i \cong e_i \cdot P$ , so that the comatrix coring  $P^\dagger \otimes_{T^\dagger} P$  is a special case of the comatrix coring studied in Section 4. In general,  $P^\dagger$  is a proper submodule of  $P^*$  and  $T^\dagger$  is a proper subalgebra of  $T$ . But we have the following remarkable result.

**Proposition 5.3.** *The map*

$$\kappa : P^\dagger \otimes_{T^\dagger} P \rightarrow P^* \otimes_T P, \kappa(\varphi \otimes_{T^\dagger} p) = \varphi \otimes_T p$$

*is an isomorphism of  $A$ -bimodules.*

*Proof.* We first define map  $\lambda : P^* \otimes P \rightarrow P^\dagger \otimes_{T^\dagger} P$  as follows: take  $\varphi \in P^*$  and  $p \in P$ . There exists  $i \in I$  such that  $p = \sigma_i(p_i)$ , and we define

$$\lambda(\varphi \otimes p) = \varphi \circ \sigma_i \circ \tau_i \otimes_{T^\dagger} p.$$

The right hand side does not depend on the choice of  $i$ : assume that  $j \in I$  is such that  $p = \sigma_j(p_j)$  for some  $p_j \in P_j$ , and take  $k \geq i, j$ . Then we have that  $p = \sigma_k(p_k)$  with  $p_k = \sigma_{ki}(p_i)$ . We compute that

$$(30) \quad \sigma_k \circ \tau_k \circ \sigma_i \circ \tau_i = \sigma_k \circ \tau_k \circ \sigma_k \circ \sigma_{ki} \circ \tau_i = \sigma_k \circ \sigma_{ki} \circ \tau_i = \sigma_i \circ \tau_i,$$

hence

$$\begin{aligned} (\varphi \circ \sigma_i \circ \tau_i) \otimes_{T^\dagger} p &= (\varphi \circ \sigma_k \circ \tau_k \circ \sigma_i \circ \tau_i) \otimes_{T^\dagger} p \\ &= (\varphi \circ \sigma_k \circ \tau_k) \otimes_{T^\dagger} (\sigma_i \circ \tau_i)(p) = (\varphi \circ \sigma_k \circ \tau_k) \otimes_{T^\dagger} p, \end{aligned}$$

and, in a similar way,

$$(\varphi \circ \sigma_j \circ \tau_j) \otimes_{T^\dagger} p = (\varphi \circ \sigma_k \circ \tau_k) \otimes_{T^\dagger} p.$$

Our next aim is to show that

$$(31) \quad \lambda(\varphi \otimes t \cdot p) = \lambda(\varphi \cdot t \otimes p),$$

for all  $\varphi \in P^*$ ,  $t \in T$ , and  $p \in P$ . There exists  $i \in I$  such that

$$p = \sigma_i(\tau_i(p)) = \sigma_i(p_i) \text{ and } t \cdot p = (\sigma_i \circ \tau_i)(t \cdot p) = (\sigma_i \circ \tau_i \circ \omega(t) \circ \sigma_i)(p_i).$$

For all  $k \geq i$ , we then also have that

$$(32) \quad p = \sigma_k(p_k) \text{ and } t \cdot p = (\sigma_k \circ \tau_k \circ \omega(t) \circ \sigma_k)(p_k).$$

For all  $p \in P$ , we have

$$\tau_i(p) = \sum z_i z_i^*(\tau_i(p)),$$

hence

$$(\omega(t) \circ \sigma_i \circ \tau_i)(p) = \sum (\omega(t) \circ \sigma_i)(z_i) z_i^*(\tau_i(p)).$$

There exists  $k \in I$  such that all  $(\omega(t) \circ \sigma_i)(z_i) \in \sigma_k(P_k)$ , or  $(\omega(t) \circ \sigma_i)(z_i) = (\sigma_k \circ \tau_k \circ \omega(t) \circ \sigma_i)(z_i)$ . Then we find for all  $p \in P$  that

$$\begin{aligned} (\omega(t) \circ \sigma_i \circ \tau_i)(p) &= \sum (\omega(t) \circ \sigma_i)(z_i) z_i^*(\tau_i(p)) \\ &= \sum (\sigma_k \circ \tau_k \circ \omega(t) \circ \sigma_i)(z_i) z_i^*(\tau_i(p)) = (\sigma_k \circ \tau_k \circ \omega(t) \circ \sigma_i \circ \tau_i)(p). \end{aligned}$$

We can take  $k \geq i$ . Using (30), we then find

$$(33) \quad \omega(t) \circ \sigma_i \circ \tau_i = \sigma_k \circ \tau_k \circ \omega(t) \circ \sigma_i \circ \tau_i = \sigma_k \circ \tau_k \circ \omega(t) \circ \sigma_k \circ \tau_k \circ \sigma_i \circ \tau_i.$$

We now compute

$$\begin{aligned} \lambda(\varphi t \otimes p) &= \varphi \circ \omega(t) \circ \sigma_i \circ \tau_i \otimes_{T^\dagger} \sigma_i(p_i) \\ &\stackrel{(33)}{=} \varphi \circ \sigma_k \circ \tau_k \circ \omega(t) \circ \sigma_k \circ \tau_k \circ \sigma_i \circ \tau_i \otimes_{T^\dagger} \sigma_i(p_i) \\ &= \varphi \circ \sigma_k \circ \tau_k \circ \omega(t) \circ \sigma_k \circ \tau_k \otimes_{T^\dagger} (\sigma_i \circ \tau_i \circ \sigma_i)(p_i) \\ &= \varphi \circ \sigma_k \circ \tau_k \circ \omega(t) \circ \sigma_k \circ \tau_k \otimes_{T^\dagger} \sigma_i(p_i) \\ &= \varphi \circ \sigma_k \circ \tau_k \circ \sigma_k \circ \tau_k \circ \omega(t) \circ \sigma_k \circ \tau_k \otimes_{T^\dagger} \sigma_k(p_k) \\ &\stackrel{(32)}{=} \varphi \circ \sigma_k \circ \tau_k \otimes_{T^\dagger} (\sigma_k \circ \tau_k \circ \omega(t) \circ \sigma_k \circ \tau_k \circ \sigma_k)(p_k) \\ &\stackrel{(32)}{=} \varphi \circ \sigma_k \circ \tau_k \otimes_{T^\dagger} tp = \lambda(\varphi \otimes tp), \end{aligned}$$

proving (31). We conclude that  $\lambda$  induces a well-defined map  $\lambda : P^* \otimes_T P \rightarrow P^\dagger \otimes_{T^\dagger} P$ . Let us finally show that  $\lambda$  is the inverse of  $\kappa$ . Take  $\varphi \in P^\dagger$  and  $p \in P$ . Then there exists  $i \in I$  such that  $\varphi = \varphi_i \circ \tau_i$  and  $p = \sigma_i(p_i)$  for some  $p_i \in P_i$ ,  $\varphi_i \in P_i^*$ . Then

$$\lambda(\kappa(\varphi \otimes_{T^\dagger} p)) = \lambda(\varphi \otimes_T p) = \varphi \circ \sigma_i \circ \tau_i \otimes_{T^\dagger} p = \varphi \otimes_{T^\dagger} (\sigma_i \circ \tau_i)(p) = \varphi \otimes_{T^\dagger} p.$$

Take  $\varphi \in P^*$ , and  $p = \sigma_i(p_i) \in P$ . Then

$$\kappa(\lambda(\varphi \otimes_T p)) = \kappa(\varphi \circ \sigma_i \circ \tau_i \otimes_{T^\dagger} p) = \varphi \circ \sigma_i \circ \tau_i \otimes_T p = \varphi \otimes_T (\sigma_i \circ \tau_i)(p) = \varphi \otimes_T p.$$

□

We will now describe the infinite comatrix coring  $P^\dagger \otimes_{T^\dagger} P$  as the colimit of a richer system. On  $I \times I$ , we define a preorder as follows.

- $(i, i) \leq (j, j)$  if  $i \leq j$  in  $I$ ;
- $(i, j) \leq (i, i)$ , for all  $i, j \in I$ ;
- $(i, j) \leq (j, j)$ , for all  $i, j \in I$ .

This preorder induces a partial order  $\leq$  on  $I \times I$ . We have a corresponding category  $\mathcal{Y}$ . If  $i \leq j$  in  $I$ , then the corresponding morphism  $(i, i) \rightarrow (j, j)$  in  $\mathcal{Y}$  is denoted by  $a_{ji}$ . The morphism  $(i, j) \rightarrow (i, i)$  is denoted by  $l_{ij}$ , and the morphism  $(i, j) \rightarrow (j, j)$  by  $r_{ij}$ . Note that we have a functor  $\xi : \mathcal{Z} \rightarrow \mathcal{Y}$ ,  $\xi(i) = (i, i)$ ,  $\xi(a_{ji}) = a_{ji}$ .

**Proposition 5.4.** *We have a functor  $\underline{F} : \mathcal{Y} \rightarrow {}_A\mathcal{M}_A$  such that  $\underline{F} \circ \xi = \underline{G}$ .*

*Proof.* For  $i, j \in I$ ,  $T_{ji} = \text{Hom}_A(M_i, M_j)$  is a  $(T_j, T_i)$ -bimodule, and we have

$$F(i, j) = P_j^* \otimes_{T_j} T_{ji} \otimes_{T_i} P_i.$$

We now define  $\underline{F}$  on the morphisms. Let  $\underline{F}(a_{ji}) = \underline{G}(a_{ji})$ ;  $\underline{F}(l_{ij})$  and  $\underline{F}(r_{ij})$  are given by

$$\underline{F}(l_{ij}) : P_j^* \otimes_{T_j} T_{ji} \otimes_{T_i} P_i \rightarrow P_i^* \otimes_{T_i} P_i, \quad \underline{F}(l_{ij})(\varphi_j \otimes_{T_j} t_{ji} \otimes_{T_i} p_i) = \varphi_j \circ t_{ji} \otimes_{T_i} p_i;$$

$$\underline{F}(r_{ij}) : P_j^* \otimes_{T_j} T_{ji} \otimes_{T_i} P_i \rightarrow P_j^* \otimes_{T_j} P_j, \quad \underline{F}(r_{ij})(\varphi_j \otimes_{T_j} t_{ji} \otimes_{T_i} p_i) = \varphi_j \otimes_{T_j} t_{ji}(p_i).$$

We have to prove that  $\underline{F}(a_{ji}) \circ \underline{F}(l_{ij}) = \underline{F}(r_{ij})$  if  $i \leq j$ . We compute easily that

$$\begin{aligned} \underline{F}(a_{ji}) \circ \underline{F}(l_{ij})(\varphi_j \otimes_{T_j} t_{ji} \otimes_{T_i} p_i) &= \varphi_j \circ t_{ji} \circ \beta(a_{ji}) \otimes_{T_j} \alpha(a_{ji})(p_i) \\ &= \varphi_j \otimes_{T_j} (t_{ji} \circ \beta(a_{ji}) \circ \alpha(a_{ji}))(p_i) \\ &= \varphi_j \otimes_{T_j} t_{ji}(p_i) = \underline{F}(r_{ij})(\varphi_j \otimes_{T_j} t_{ji} \otimes_{T_i} p_i). \end{aligned}$$

In a similar way, we prove that  $\underline{F}(a_{ji}) \circ \underline{F}(r_{ij}) = \underline{F}(l_{ij})$  if  $i \leq j$ . All other verifications are easy.  $\square$

**Proposition 5.5.**  $\text{colim } \underline{F} = (P^\dagger \otimes_{T^\dagger} P, f)$  with

$$f_{ij} = g_i \circ \underline{F}(l_{ij}) = g_j \circ \underline{G}(r_{ij}),$$

for all  $i, j \in I$ .

*Proof.* It is easy to show that  $(P^\dagger \otimes_{T^\dagger} P, f)$  is a cocone on  $\underline{F}$ . If  $(M, m)$  is another cocone on  $\underline{F}$ , then we have a cocone  $(M, n)$  on  $\underline{G}$ , with  $n_i = m_{(i,i)}$ . We then have an  $A$ -bimodule map  $f : P^\dagger \otimes_{T^\dagger} P$ , and it is straightforward to show that it satisfies the necessary requirements.  $\square$

## 6. SPLIT DIRECT SYSTEMS OF GALOIS COMODULES

Let  $A$  be a  $k$ -algebra (with unit), and  $\mathcal{C}$  an  $A$ -coring. By [6, 18.12] the category  $\mathcal{M}^{\mathcal{C}}$  contains direct sums and cokernels. Consequently  $\mathcal{M}^{\mathcal{C}}$  contains colimits, so in particular directed limits. Moreover, the forgetful functor  $\omega : \mathcal{M}^{\mathcal{C}} \rightarrow \mathcal{M}_A$  has a right adjoint, so it preserves colimits (see for example [14, Sec. V.5]). Hence we can apply the results of Section 5 in the situation where  $\mathcal{A} = \mathcal{M}^{\mathcal{C}}$ .

Now we consider a split direct system  $\underline{M}^s : \mathcal{Z} \rightarrow \mathcal{M}_{\text{fgp}}^{\mathcal{C}}$ ,  $\underline{M}^s(i) = M_i$  and  $\underline{M}^s(a_{ji}) = (\mu_{ji}, \nu_{ij})$ . Here  $\mathcal{M}_{\text{fgp}}^{\mathcal{C}}$  denotes the category of right  $\mathcal{C}$ -comodules that are finitely generated and projective as right  $A$ -module. We can compute  $\text{colim } \underline{M} = (M, \mu)$  in  $\mathcal{M}^{\mathcal{C}}$ , and from Proposition 3.1 we know that there exist left inverses  $\nu_i$  of the  $\mu_i$ . As in Section 5, let  $T_i = \text{End}^{\mathcal{C}}(M_i)$ , and  $(T^\dagger, t) = \text{colim } T$ . We have the associated comatrix coring  $\mathcal{G} = M^\dagger \otimes_{T^\dagger} M$ . For every  $i \in I$ , we have a morphism of corings

$$\text{can}_i : G(i) = M_i^* \otimes_{T_i} M_i \rightarrow \mathcal{C}, \quad \text{can}_i(\varphi_i \otimes_{T_i} m_i) = \varphi_i(m_{i[0]})m_{i[1]}.$$

**Lemma 6.1.**  $(\mathcal{C}, \text{can})$  is a cocone on  $\underline{G} : \mathcal{Z} \rightarrow {}_A\mathcal{M}_A$

*Proof.* For  $i \leq j$  in  $I$ ,  $m_i \in M_i$  and  $\varphi_i \in M_i^*$ , we calculate that

$$\begin{aligned} (\text{can}_j \circ \underline{G}(a_{ji}))(\varphi_i \otimes_{T_i} m_i) &= \text{can}_j(\varphi_i \circ \nu_{ij} \otimes_{T_j} \mu_{ji}(m_i)) \\ &= (\varphi_i \circ \nu_{ij})(\mu_{ji}(m_i)_{[0]})\mu_{ji}(m_i)_{[1]} = \varphi_i(\tau_{ij}(\mu_{ji}(m_{i[0]})))m_{i[1]} \\ &= \varphi_i(m_{i[0]})m_{i[1]} = \text{can}_i(\varphi_i \otimes_{T_i} m_i), \end{aligned}$$

where we used the fact that  $\mu_{ji}$  is right  $\mathcal{C}$ -colinear.  $\square$

**Proposition 6.2.** *There exists a unique morphism of corings*

$$\text{can} : \mathcal{G} = M^\dagger \otimes_{T^\dagger} M \rightarrow \mathcal{C}$$

such that

$$(34) \quad \text{can}(\varphi_i \circ \nu_i \otimes_{T^\dagger} \mu_i(m_i)) = \varphi_i(m_{i[0]})m_{i[1]},$$

for all  $i \in I$ ,  $m_i \in M_i$  and  $\varphi_i \in M_i^*$ .

*Proof.* We have seen in Proposition 4.13 that  $\text{colim} \underline{\mathcal{G}} = (\mathcal{G}, g)$ . It follows from Lemma 6.1 and the universal property of colimits that there exists a unique  $(A, A)$ -bimodule map  $\text{can} : \mathcal{G} \rightarrow \mathcal{C}$  satisfying (34). The proof is finished if we can show that  $\text{can}$  is a map of corings. As before, let  $E_i = \sum z_i \otimes_A z_i^*$  be a finite dual basis of  $M$  as a right  $A$ -module. For all  $m_i \in M_i$ , we have that  $\sum z_i z_i^*(m_i) = m_i$ . Since  $\rho_i$  is right  $A$ -linear, we have

$$\sum z_{i[0]} \otimes_A z_{i[1]} z_i^*(m_i) = m_{i[0]} \otimes_A m_{i[1]},$$

hence, for all  $\varphi_i \in M_i$ :

$$(35) \quad \sum \varphi_i(z_{i[0]}) z_{i[1]} z_i^*(m_i) = \varphi_i(m_{i[0]}) m_{i[1]}.$$

Then we compute

$$\begin{aligned} (\text{can} \otimes_A \text{can}) \Delta(\varphi_i \circ \nu_i \otimes_{T^\dagger} \mu_i(m_i)) &= \sum \varphi_i(z_{i[0]}) z_{i[1]} \otimes_A z_i^*(p_{i[0]}) p_{i[1]} \\ &= \sum \varphi_i(z_{i[0]}) z_{i[1]} z_i^*(p_{i[0]}) \otimes_A p_{i[1]} \stackrel{(35)}{=} \varphi_i(p_{i[0]}) p_{i[1]} \otimes_A p_{i[2]} \\ &= \Delta(\varphi_i(p_{i[0]}) p_{i[1]}) = \Delta(\text{can}(\varphi_i \circ \nu_i \otimes_{T^\dagger} \mu_i(m_i))). \end{aligned}$$

Finally

$$\begin{aligned} \varepsilon(\text{can}(\varphi_i \circ \nu_i \otimes_{T^\dagger} \mu_i(m_i))) &= \varepsilon(\varphi_i(m_{i[0]}) m_{i[1]}) = \varphi_i(m_{i[0]}) \varepsilon(m_{i[1]}) \\ &= \varphi_i(m_{i[0]} \varepsilon(m_{i[1]})) = \varphi_i(m_i) = \varepsilon(\varphi_i \circ \nu_i \otimes_{T^\dagger} \mu_i(m_i)). \end{aligned}$$

□

We call  $\underline{M}^s$  a *split direct system of Galois  $\mathcal{C}$ -comodules* if  $\text{can} : \mathcal{G} \rightarrow \mathcal{D}$  is an isomorphism of corings. In this situation, we have that the categories  $\mathcal{M}^{\mathcal{G}}$  and  $\mathcal{M}^{\mathcal{C}}$  are isomorphic. From Theorem 1.3, we then immediately obtain the following result:

**Theorem 6.3.** *Let  $\mathcal{C}$  be an  $A$ -coring, and  $\underline{M}^s$  a split direct system of Galois  $\mathcal{C}$ -comodules. Let  $\text{colim} \underline{M} = (M, \mu)$ ,  $\text{colim} \text{End}^{\mathcal{C}}(\underline{M}) = (T^\dagger, \rho)$  and  $\mathcal{G} = M^\dagger \otimes_{T^\dagger} M$  the associated comatrix coring. If  $M$  is faithfully flat as a left  $T^\dagger$ -module, then the categories  $\mathcal{M}_{T^\dagger}$  and  $\mathcal{M}^{\mathcal{C}}$  are equivalent.*

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